University of Calicut

B.Sc. Mathematics Honours — CUFYUGP 2024

Semester I — MAT1CJ101

Differential Calculus

A Student Study Guide

Build intuition. Understand deeply. Prepare for exams.

Based on:

G. B. Thomas Jr. and R. L. Finney,
Calculus and Analytic Geometry, 9th Ed., Pearson, 1995

Academic Year 2024–25

Preface

This book is a study guide for **MAT1CJ101 — Differential Calculus**, a core course in the B.Sc. Mathematics Honours programme at colleges affiliated with the University of Calicut under the CUFYUGP 2024 regulations.

Who is this guide for? You are a first-year Mathematics Honours student. You may have seen some calculus in school, but this course takes it much further. We explain every idea from first principles, with intuition before formalism and concrete examples before abstractions.

How to read this guide. Each chapter covers one module of the syllabus. Within each chapter, topics follow a consistent flow:

1. **Intuition box** (green) — the big idea in plain language, before any symbols.
2. **Definition box** (blue) — the precise mathematical statement.
3. **Proof box** (teal) — a complete proof of the theorem or rule.
4. **Worked Example box** (orange) — a fully solved example with every step shown.
5. **Exam Tip box** (red) — what the university exam specifically tests on this topic.
6. **Key Takeaway box** (purple) — the one sentence to carry forward.
7. **Warning box** (yellow) — a common mistake to avoid.

Not every topic uses all seven elements; short topics may only have a definition and an example.

Exercises. Each chapter ends with a set of exercises in four levels:

- **Level 1 — Recall and Understand:** straightforward definitions and simple computations.
- **Level 2 — Apply:** standard exam-style problems requiring the techniques from the chapter.
- **Level 3 — Prove:** problems asking you to write a proof; proofs in this course are typically short (3–8 lines).
- **Level 4 — Challenge:** harder problems that combine ideas or require a non-obvious step.

Answers are *not* given after each exercise. Hints, final answers, and key steps for all exercises appear in the **Hints and Answers** section at the back of the book.

Previous year question paper. The first batch of CUFYUGP 2024 students sat the MAT1CJ101 external exam in October 2024. That paper is reproduced in its entirety in the appendix. Throughout the main text, worked examples drawn from that paper are tagged [Oct 2024].

Exam pattern.

Section	Marks per question	Maximum marks
A (short answer)	3	Ceiling 24
B (medium answer)	6	Ceiling 36
C (long answer)	10	Any one of two options
External total		70
Internal		30

The external exam covers **Chapters 1–4** only (Modules I–IV), with a minimum of 15 marks from each chapter. **Chapter 5** (Module V) is assessed internally only.

Notation. We write $f'(x)$ for the derivative, and $f''(x)$ for the second derivative. The symbols $\lim_{x\to a}$, $\lim_{x\to a^+}$, and $\lim_{x\to a^-}$ denote two-sided, right-hand, and left-hand limits. $\mathbb{R}$ is the set of all real numbers. All intervals are subsets of $\mathbb{R}$ unless stated otherwise.

Contents

Chapter 1

Functions and Limits

Module I · 12 Hours · Preliminary §3–4; §1.1, §1.2, §1.4 of Thomas & Finney (9th Ed.)

What you will learn in this chapter.

- What a function is: domain, range, and how functions can be combined.
- How to shift, reflect, and scale the graph of a function.
- What a *limit* is and why it is the central concept of calculus.
- How to compute limits using algebra: factoring, rationalizing, and the Squeeze Theorem.
- The behaviour of functions as x grows without bound (limits at infinity).

1.1 Functions: Domain, Range, and Algebraic Combinations

1.1.1 Definition and Notation

A function is a rule that assigns to each input exactly one output.

Definition

A **function** f from a set D to a set Y is a rule that assigns to each element $x \in D$ a unique element $f(x) \in Y$. The set D is the **domain** of f. The **range** of f is $\{f(x) : x \in D\} \subseteq Y$.

Intuition

Think of a function as a machine: you put in a number x, and it produces a number $f(x)$. The domain is the set of all inputs the machine can handle. The range is the set of all outputs the machine can produce.

Example 1.1 — Domain from an explicit formula

Find the domain of $f(x) = \dfrac{\sqrt{x-2}}{x-5}$.

Solution. We need two conditions simultaneously:

1. $\sqrt{x-2}$ requires $x-2 \geq 0$, i.e. $x \geq 2$.
2. The denominator $x-5 \neq 0$, i.e. $x \neq 5$.

So the domain is $\{x \in \mathbb{R} : x \geq 2 \text{ and } x \neq 5\} = [2,5) \cup (5,\infty)$.

Example 1.2 — Domain with multiple restrictions

Find the domain of $g(x) = \dfrac{1}{\sqrt{4-x^2}}$.

Solution. We need $4-x^2 > 0$ (strictly positive because it appears under a square root in a denominator). $4-x^2 > 0 \Leftrightarrow x^2 < 4 \Leftrightarrow -2 < x < 2$. Domain: $(-2,2)$.

Example 1.3 — Range of a function

Find the range of $f(x) = x^2 + 1$.

Solution. For any real x, we have $x^2 \geq 0$, so $f(x) = x^2 + 1 \geq 1$. Conversely, for any $y \geq 1$, we can solve $x^2 + 1 = y$ to get $x = \pm\sqrt{y-1}$, which is real. So the range is $[1,\infty)$.

Example 1.4 — Range of a rational function

Find the range of $h(x) = \dfrac{1}{1+x^2}$.

Solution. Since $x^2 \geq 0$, we have $1 + x^2 \geq 1$, so $h(x) = \frac{1}{1+x^2} \leq 1$. Also $1 + x^2 > 0$ always, so $h(x) > 0$. As $x \to \pm\infty$, $h(x) \to 0$ (but never equals 0). At $x = 0$, $h(0) = 1$. So the range is $(0,1]$.

1.1.2 Algebraic Combinations of Functions

Definition

Given functions f and g with domains D_f and D_g, we define:

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x), && \text{domain } D_f \cap D_g \\
(f-g)(x) &= f(x) - g(x), && \text{domain } D_f \cap D_g \\
(fg)(x) &= f(x) \cdot g(x), && \text{domain } D_f \cap D_g \\
(f/g)(x) &= f(x)/g(x), && \text{domain } \{x \in D_f \cap D_g : g(x) \neq 0\}
\end{aligned}$$

1.2 Composition of Functions

Definition

The **composition** of f with g is the function $(f \circ g)(x) = f(g(x))$, defined for all x in the domain of g such that $g(x)$ is in the domain of f.

Intuition

Composition chains two machines together: x goes in, g processes it to $g(x)$, then f processes that to $f(g(x))$. The output of the first machine must be a valid input for the second.

Warning

The compositions $f \circ g$ and $g \circ f$ are generally **different**.

Example 1.5 — Computing compositions

Let $f(x) = x^2$ and $g(x) = x + 1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution.

$$(f \circ g)(x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$$
$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1$$

These are different polynomials, confirming order matters.

Example 1.6 — Domain of a composition

Let $f(x) = \sqrt{x}$ (domain $[0, \infty)$) and $g(x) = 1 - x$ (domain $\mathbb{R}$). Find $(f \circ g)(x)$ and its domain.

Solution. $(f \circ g)(x) = f(g(x)) = f(1-x) = \sqrt{1-x}$.
For this to be defined, we need $g(x) = 1 - x \geq 0$, i.e. $x \leq 1$. Domain of $f \circ g$: $(-\infty, 1]$.

Example 1.7 — Decomposing a composite function

Write $h(x) = (x^2 + 1)^{1/3}$ as $f \circ g$ for suitable f and g.

Solution. Let $g(x) = x^2 + 1$ (the inner function, computed first) and $f(u) = u^{1/3}$ (the outer function). Then $(f \circ g)(x) = f(g(x)) = (x^2+1)^{1/3} = h(x)$. ✓
Note: the decomposition is not unique. We could also take $g(x) = x^2$ and $f(u) = (u+1)^{1/3}$.

Example 1.8 — Three-layer composition

Find $(f \circ g \circ h)(x)$ where $f(u) = \sqrt{u}$, $g(v) = v + 4$, $h(x) = x^2$.

Solution. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f(x^2 + 4) = \sqrt{x^2 + 4}$. Since $x^2 + 4 \geq 4 > 0$ for all x, the domain is all of $\mathbb{R}$.

1.3 Even, Odd, and Periodic Functions

Definition

A function f is:

- **even** if $f(-x) = f(x)$ for all x in its domain. (Graph is symmetric about the y-axis.)
- **odd** if $f(-x) = -f(x)$ for all x in its domain. (Graph has $180°$ rotational symmetry about the origin.)
- **periodic** with period $T > 0$ if $f(x + T) = f(x)$ for all x.

Example 1.9 — Classifying functions as even, odd, or neither

Classify each function: (a) $f(x) = x^3 - x$ (b) $g(x) = x^2 + 1$ (c) $h(x) = x + 1$ (d) $k(x) = |x|$

Solution.

(a) $f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$. **Odd.**
(b) $g(-x) = (-x)^2 + 1 = x^2 + 1 = g(x)$. **Even.**
(c) $h(-x) = -x + 1 \neq h(x)$ and $\neq -h(x)$ (unless $x = 0$). **Neither.**
(d) $k(-x) = |-x| = |x| = k(x)$. **Even.**

Example 1.10 — Products and sums of even/odd functions

Let f be even and g be odd. Show that fg is odd.

Proof. $(fg)(-x) = f(-x) \cdot g(-x) = f(x) \cdot (-g(x)) = -f(x)g(x) = -(fg)(x)$. So fg is odd. □

1.4 Shifting and Scaling Graphs

Intuition

You do not need to recompute a function from scratch if you know the graph of a simpler function. Shifts and scalings transform the known graph into the new one.

Definition

Starting from the graph of $y = f(x)$:

Operation	Effect on graph
$y = f(x) + c$, $c > 0$	Shift **up** by c
$y = f(x) - c$, $c > 0$	Shift **down** by c
$y = f(x - d)$, $d > 0$	Shift **right** by d
$y = f(x + d)$, $d > 0$	Shift **left** by d
$y = -f(x)$	Reflect about the x-axis
$y = f(-x)$	Reflect about the y-axis
$y = cf(x)$, $c > 1$	Vertical **stretch** by factor c
$y = cf(x)$, $0 < c < 1$	Vertical **compression** by factor c
$y = f(x/c)$, $c > 1$	Horizontal **stretch** by factor c
$y = f(cx)$, $c > 1$	Horizontal **compression** by factor c

Example 1.11 — Vertical shift

Describe the graph of $y = x^2 + 3$ in terms of $y = x^2$.
Solution. The graph of $y = x^2 + 3$ is the parabola $y = x^2$ shifted **up by 3 units**. Vertex moves from $(0, 0)$ to $(0, 3)$.

Example 1.12 — Horizontal shift

Describe $y = (x - 2)^2$.
Solution. Replace x by $x - 2$: graph of $y = x^2$ shifted **right by 2**. Vertex at $(2, 0)$.

Example 1.13 — Reflection about the x-axis

Describe $y = -\sqrt{x}$.
Solution. The graph of $y = \sqrt{x}$ (in the first quadrant) is reflected about the x-axis. The resulting curve lies in the fourth quadrant: for $x \geq 0$, $y = -\sqrt{x} \leq 0$.

Example 1.14 — Combined transformation

Describe $y = -2(x + 1)^2 + 3$ starting from $y = x^2$.

Solution. Apply transformations in order:
1. Start: $y = x^2$.
2. Replace x by $x + 1$: shift **left 1**: $y = (x + 1)^2$.
3. Multiply by 2: vertical **stretch by 2**: $y = 2(x + 1)^2$.
4. Multiply by -1: reflect about x-axis: $y = -2(x + 1)^2$.
5. Add 3: shift **up 3**: $y = -2(x + 1)^2 + 3$.

Result: downward-opening parabola with vertex at $(-1, 3)$.

Example 1.15 — Graph of $y = 1 + \sqrt{x - 2}$ (Oct 2024 Q11)

Sketch $y = 1 + \sqrt{x - 2}$ and state the domain.

Solution. Start with $y = \sqrt{x}$ (domain $[0, \infty)$, range $[0, \infty)$).

Step 1: Replace x by $x-2$: shift right $2 \to y = \sqrt{x-2}$ (domain $[2, \infty)$).
Step 2: Add 1: shift up $1 \to y = 1 + \sqrt{x-2}$.
Domain: $x \geq 2$, i.e. $[2, \infty)$. At $x = 2$: $y = 1$. As $x \to \infty$: $y \to \infty$. The graph starts at $(2, 1)$ and rises to the right, looking like the square root curve shifted.

1.5 Rates of Change and the Limit Concept

1.5.1 Average Rate of Change

Definition

The **average rate of change** of f over $[x_1, x_2]$ is:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta f}{\Delta x}.$$

Geometrically, this is the *slope of the secant line* through $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Example 1.16 — Average rate of change as $h \to 0$

Find the average rate of change of $f(x) = x^2$ on the interval $[1, 1+h]$ for $h \neq 0$, and find the limit as $h \to 0$.

Solution.

$$\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h.$$

As $h \to 0$, this approaches **2**. This is the slope of the tangent to the parabola at $x = 1$.

Example 1.17 — Speed as average rate of change

A car travels $s(t) = 4t^2$ metres in t seconds. Find its average speed over $[2, 2.1]$.

Solution. $\frac{s(2.1)-s(2)}{2.1-2} = \frac{4(2.1)^2-4(4)}{0.1} = \frac{4(4.41)-16}{0.1} = \frac{17.64-16}{0.1} = \frac{1.64}{0.1} = 16.4$ m/s.

1.5.2 The Idea of a Limit

The limit of $f(x)$ as $x \to a$ is the value that $f(x)$ approaches as x gets arbitrarily close to a, *but does not equal* a.

Definition

We write $\lim_{x \to a} f(x) = L$ and say **the limit of $f(x)$ as x approaches a is L**, if $f(x)$ can be made as close to L as desired by taking x sufficiently close to (but not equal to) a.

Example 1.18 — Limit of a removable discontinuity

Evaluate $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$.

Solution. The function $f(x) = (x^2-4)/(x-2)$ is *not defined* at $x = 2$ (division by zero). But for $x \neq 2$:

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2.$$

As $x \to 2$, $x+2 \to 4$. Therefore $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = 4$.

Note: The function f is not defined at $x = 2$, but the limit still exists and equals 4.

1.6 One-Sided Limits

Definition

- The **right-hand limit** $\lim\limits_{x\to a^+} f(x) = L$ means $f(x) \to L$ as $x \to a$ from the right $(x > a)$.
- The **left-hand limit** $\lim\limits_{x\to a^-} f(x) = L$ means $f(x) \to L$ as $x \to a$ from the left $(x < a)$.

The two-sided limit $\lim_{x\to a} f(x) = L$ exists if and only if both one-sided limits exist and are equal:

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

Example 1.19 — One-sided limits of $|x|/x$

Evaluate $\lim_{x\to 0^+} \frac{|x|}{x}$ and $\lim_{x\to 0^-} \frac{|x|}{x}$.

Solution. For $x > 0$: $|x| = x$, so $|x|/x = x/x = 1$. Hence $\lim_{x\to 0^+} \frac{|x|}{x} = 1$.
For $x < 0$: $|x| = -x$, so $|x|/x = -x/x = -1$. Hence $\lim_{x\to 0^-} \frac{|x|}{x} = -1$.
Since the one-sided limits are different, $\lim_{x\to 0} \frac{|x|}{x}$ **does not exist**.

Example 1.20 — Limit of a piecewise function

Let $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2x-1 & \text{if } x \geq 1 \end{cases}$. Find $\lim_{x\to 1} f(x)$.

Solution. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$.
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x-1) = 2(1) - 1 = 1$.
Both one-sided limits equal 1, so $\lim_{x\to 1} f(x) = 1$. (Note: $f(1) = 2(1) - 1 = 1$ as well, so f is actually continuous at $x = 1$.)

Example 1.21 — Limit does not exist (one-sided limits differ)

Let $g(x) = \begin{cases} 3 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases}$. Find $\lim_{x\to 0} g(x)$.

Solution. $\lim_{x\to 0^-} g(x) = 3$ (approaching from the left, $g(x) = 3$).
$\lim_{x\to 0^+} g(x) = -1$ (approaching from the right, $g(x) = -1$).
The limits disagree, so $\lim_{x\to 0} g(x)$ **does not exist**.

1.7 Limit Laws

Theorem 1.1 (Limit Laws). *Suppose* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then:*

$$\begin{aligned}
&\textit{(Sum)} && \lim_{x\to a}[f(x) + g(x)] = L + M \\
&\textit{(Difference)} && \lim_{x\to a}[f(x) - g(x)] = L - M \\
&\textit{(Product)} && \lim_{x\to a}[f(x) \cdot g(x)] = L \cdot M \\
&\textit{(Quotient)} && \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad (M \neq 0) \\
&\textit{(Power)} && \lim_{x\to a}[f(x)]^n = L^n \quad (n \textit{ a positive integer}) \\
&\textit{(Root)} && \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \quad (L > 0 \textit{ if } n \textit{ is even})
\end{aligned}$$

Proof of the Sum Law

Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}.$$

Since $\lim_{x\to a} g(x) = M$, there exists $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}.$$

Set $\delta = \min(\delta_1, \delta_2)$. For $0 < |x - a| < \delta$:

$$|[f(x) + g(x)] - [L + M]| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(We used the triangle inequality $|A + B| \leq |A| + |B|$.) This proves the Sum Law. □

Example 1.22 — Using limit laws to compute limits

Evaluate $\lim_{x\to 3}(x^2 + 2x - 1)$.

Solution. Applying the limit laws step by step:

$$\begin{aligned}\lim_{x\to 3}(x^2+2x-1) &= \lim_{x\to 3} x^2 + \lim_{x\to 3} 2x - \lim_{x\to 3} 1 && \text{(Sum/Difference Law)}\\ &= \left(\lim_{x\to 3} x\right)^2 + 2\lim_{x\to 3} x - 1 && \text{(Power and Constant-Multiple Laws)}\\ &= 3^2 + 2(3) - 1 = 9 + 6 - 1 = \mathbf{14}.\end{aligned}$$

Example 1.23 — Limits of rational functions by substitution

Evaluate $\lim_{x\to -2} \frac{x^3+8}{x^2-4}$ if the limit exists.

Solution. At $x = -2$: numerator $= (-2)^3 + 8 = -8 + 8 = 0$ and denominator $= 4 - 4 = 0$. This is a $0/0$ indeterminate form; direct substitution fails.
Factor:

$$\frac{x^3+8}{x^2-4} = \frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)} = \frac{x^2-2x+4}{x-2} \quad (x \neq -2).$$

As $x \to -2$:

$$\lim_{x\to -2} \frac{x^2-2x+4}{x-2} = \frac{4+4+4}{-4} = \frac{12}{-4} = -3.$$

1.8 Evaluating Limits: Algebraic Techniques

1.8.1 Factoring

When a limit has the $0/0$ form, try to factor and cancel.

Example 1.24 — Factoring a difference of cubes

$\lim_{x\to 2} \frac{x^3-8}{x-2}$.

Solution. Direct substitution gives $0/0$. Factor: $x^3 - 8 = (x-2)(x^2+2x+4)$.

$$\lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x\to 2}(x^2+2x+4) = 4+4+4 = \mathbf{12}.$$

Example 1.25 — Factoring a quadratic

$\lim_{x\to 3} \frac{3x^2-x-24}{x-3}$.

Solution. Check: at $x = 3$, numerator $= 27 - 3 - 24 = 0$. Factor $3x^2 - x - 24 = (3x+8)(x-3)$:

$$\lim_{x\to 3} \frac{(3x+8)(x-3)}{x-3} = \lim_{x\to 3}(3x+8) = 17.$$

1.8.2 Rationalizing

When a square root appears in a 0/0 form, multiply by the *conjugate*.

Example 1.26 — Rationalizing (Oct 2024 Q2)

Evaluate $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$.

Solution. Direct substitution gives 0/0. Multiply numerator and denominator by $\sqrt{x+2}+\sqrt{2}$ (the conjugate):

$$\frac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}} = \frac{(x+2)-2}{x(\sqrt{x+2}+\sqrt{2})} = \frac{x}{x(\sqrt{x+2}+\sqrt{2})} = \frac{1}{\sqrt{x+2}+\sqrt{2}}.$$

As $x \to 0$:

$$\lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

Example 1.27 — Rationalizing with a square root in the numerator

$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$.

Solution. Multiply by $\frac{\sqrt{x}+2}{\sqrt{x}+2}$:

$$\frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}.$$

As $x \to 4$: $\frac{1}{\sqrt{4}+2} = \frac{1}{4}$.

Example 1.28 — A cube-root limit

$\lim_{x\to 1}\frac{x^{1/3}-1}{x-1}$.

Solution. Let $u = x^{1/3}$, so $x = u^3$ and $x \to 1 \Leftrightarrow u \to 1$.

$$\lim_{u\to 1}\frac{u-1}{u^3-1} = \lim_{u\to 1}\frac{u-1}{(u-1)(u^2+u+1)} = \lim_{u\to 1}\frac{1}{u^2+u+1} = \frac{1}{3}.$$

1.9 The Squeeze Theorem

Theorem 1.2 (Squeeze Theorem). *Suppose $g(x) \le f(x) \le h(x)$ for all x near a (except possibly at a itself), and*

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

Then $\lim_{x\to a} f(x) = L$.

Proof of the Squeeze Theorem

Let $\varepsilon > 0$. Since $\lim_{x\to a} g(x) = L$, there exists $\delta_1 > 0$ with $|x-a| < \delta_1 \Rightarrow |g(x) - L| < \varepsilon$. Since $\lim_{x\to a} h(x) = L$, there exists $\delta_2 > 0$ with $|x-a| < \delta_2 \Rightarrow |h(x) - L| < \varepsilon$.
Set $\delta = \min(\delta_1, \delta_2)$. For $0 < |x-a| < \delta$:

$$L - \varepsilon < g(x) \le f(x) \le h(x) < L + \varepsilon.$$

So $L - \varepsilon < f(x) < L + \varepsilon$, i.e. $|f(x) - L| < \varepsilon$. $\square$

Intuition

If you are squeezed between two people walking toward the same door, you must also reach that door. The "squeezing" functions g and h push f to the same limit L.

Example 1.29 — Classic Squeeze Theorem application

Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Solution. Since $-1 \le \sin(1/x) \le 1$ for all $x \ne 0$:

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2.$$

Both $-x^2$ and x^2 tend to 0 as $x \to 0$. By the Squeeze Theorem:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Example 1.30 — Another squeeze

Show $\lim_{x\to 0} x\cos(1/x) = 0$.

Solution. Since $|\cos(1/x)| \le 1$ for $x \ne 0$:

$$-|x| \le x\cos(1/x) \le |x|.$$

Since $|x| \to 0$ as $x \to 0$, the Squeeze Theorem gives the limit 0.

Exam Tip

The Squeeze Theorem is the standard tool whenever the function is a product of a term going to zero (like x^2 or x) and a bounded oscillating term (like $\sin(1/x)$ or $\cos(1/x)$). Always state the bounding inequality and the limit of the bounds explicitly.

1.10 Infinite Limits and Limits at Infinity

1.10.1 Infinite Limits (Vertical Asymptotes)

Definition

We write $\lim_{x \to a} f(x) = +\infty$ if $f(x)$ grows without bound as $x \to a$. Similarly, $\lim_{x \to a} f(x) = -\infty$ means $f(x)$ decreases without bound. The line $x = a$ is called a **vertical asymptote** of f.

Example 1.31 — One-sided infinite limits

Evaluate $\lim_{x \to 0^+} \dfrac{1}{x}$ and $\lim_{x \to 0^-} \dfrac{1}{x}$.

Solution. For $x > 0$ small: $1/x$ is large positive. So $\lim_{x \to 0^+} \dfrac{1}{x} = +\infty$.

For $x < 0$ small (negative): $1/x$ is large negative. So $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$.
The two-sided limit does not exist; $x = 0$ is a vertical asymptote.

Example 1.32 — Sign analysis for infinite limits

Evaluate $\displaystyle\lim_{x \to 2} \frac{x+3}{(x-2)^2}$.

Solution. As $x \to 2$: numerator $\to 5 > 0$; denominator $(x-2)^2 \to 0^+$ (always positive). So the limit is $+\infty$ (from both sides, since the square makes the denominator always positive).

Example 1.33 — Vertical asymptote with sign change

Evaluate the one-sided limits of $\dfrac{x-1}{x^2-4}$ as $x \to 2$.

Solution. At $x = 2$: numerator $\to 1 > 0$; denominator $\to 0$.

- As $x \to 2^+$: $x - 2 > 0$, so $(x^2 - 4) = (x-2)(x+2) > 0$. Fraction $\to +\infty$.
- As $x \to 2^-$: $x - 2 < 0$, so $(x^2 - 4) < 0$. Fraction $\to -\infty$.

1.10.2 Limits at Infinity (Horizontal Asymptotes)

Definition

We write $\lim_{x \to +\infty} f(x) = L$ if $f(x)$ can be made arbitrarily close to L by taking x sufficiently large. The line $y = L$ is a **horizontal asymptote** of f.

Intuition

For rational functions, the limit at infinity is determined by the *highest-degree terms*. Divide every term by the highest power of x in the denominator, then let $x \to \infty$.

Example 1.34 — Horizontal asymptote of a rational function

$\lim_{x \to \infty} \frac{2x+3}{x-1}$.

Solution. Divide numerator and denominator by x:

$$\frac{2x+3}{x-1} = \frac{2+3/x}{1-1/x}.$$

As $x \to \infty$: $3/x \to 0$ and $1/x \to 0$. Limit $= \frac{2+0}{1-0} = \mathbf{2}$.

Example 1.35 — Rational function, degree of numerator = degree of denominator

$\lim_{x \to \infty} \frac{3x^2-2x+1}{2x^2+x-3}$.

Solution. Divide by x^2:

$$\frac{3-2/x+1/x^2}{2+1/x-3/x^2} \to \frac{3}{2} \quad \text{as } x \to \infty.$$

Horizontal asymptote: $y = 3/2$.

Example 1.36 — Limit at infinity with a square root

$\lim_{x \to +\infty} \frac{x}{\sqrt{x^2+1}}$.

Solution. Divide numerator and denominator by x (positive for $x > 0$; use $x = \sqrt{x^2}$):

$$\frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{1+1/x^2}} \to \frac{1}{\sqrt{1+0}} = 1.$$

As $x \to -\infty$ (now $x < 0$, so $x = -\sqrt{x^2}$):

$$\frac{x}{\sqrt{x^2+1}} = \frac{-\sqrt{x^2}}{\sqrt{x^2+1}} = -\frac{1}{\sqrt{1+1/x^2}} \to -1.$$

So $y = 1$ is a HA as $x \to +\infty$ and $y = -1$ as $x \to -\infty$.

Example 1.37 — Limit at infinity of $(\sin x)/x$

$\lim_{x \to \infty} \frac{\sin x}{x}$.

Solution. Since $-1 \leq \sin x \leq 1$:

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}.$$

Both bounds tend to 0 as $x \to \infty$. By the Squeeze Theorem: $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

1.11 Chapter Summary

Concept	**Key Fact**
Domain	Largest set of x for which $f(x)$ is defined
Composition	$(f \circ g)(x) = f(g(x))$; domain: $g(x)$ must lie in domain of f
Even / Odd	$f(-x) = f(x)$ (even); $f(-x) = -f(x)$ (odd)
Graph shift	$f(x-d)$: right d; $f(x)+c$: up c; $-f(x)$: reflect x-axis
Two-sided limit	Exists $\Leftrightarrow$ both one-sided limits exist and are equal
Limit laws	Sum, product, quotient laws (quotient requires $M \neq 0$)
0/0 form	Try factoring, rationalizing, or substitution
Squeeze Theorem	$g \le f \le h$, $g, h \to L \Rightarrow f \to L$
Infinite limit	$f(x) \to \pm\infty$ as $x \to a$: vertical asymptote $x = a$
Limit at infinity	Divide by dominant power; $y = L$: horizontal asymptote

Key Takeaway

The limit $\lim_{x\to a} f(x)$ describes the *behaviour near* a, not the value *at* a. The function need not even be defined at a.

1.12 Exercises

Level 1 — Recall and Understand

1. Find the domain of $f(x) = \dfrac{\sqrt{x-2}}{x+1}$.
2. Find the domain of $g(x) = \dfrac{1}{\sqrt{4-x^2}}$.
3. Let $f(x) = x^2$ and $g(x) = 2x+1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and state their domains.
4. Determine whether $f(x) = x^4 - 2x^2$ is even, odd, or neither.
5. Describe, in words, how the graph of $y = -\sqrt{x+3}$ is obtained from the graph of $y = \sqrt{x}$.

Level 2 — Apply

6. Evaluate $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$.
7. Evaluate $\lim_{x\to -1} \dfrac{x^3+1}{x+1}$.

8. Evaluate $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$ by rationalizing.
9. Evaluate $\lim\limits_{x\to 0} \dfrac{|x|}{x}$, or explain why it does not exist.
10. Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2-2x+1}{2x^2+x-3}$.
11. Evaluate $\lim\limits_{x\to\infty} \dfrac{x+1}{\sqrt{x^2+x}}$.
12. Show that $\lim\limits_{x\to 0} x^2 \cos\left(\dfrac{1}{x^2}\right) = 0$ using the Squeeze Theorem.
13. A particle's position is $s(t) = 2t^2 - t$. Find its average velocity over $[1, 1+h]$ and the limit as $h \to 0$.
14. Find the one-sided limits of $f(x) = \dfrac{x-1}{(x-1)^2}$ as $x \to 1$, or explain why they are infinite.
15. Evaluate $\lim\limits_{x\to\infty} \dfrac{\sin(2x)}{x}$.

Level 3 — Prove

16. Prove from the ε-δ definition of limit that $\lim_{x\to 2}(3x+1) = 7$.
17. Prove that if f is an even function and g is an odd function, then $f \circ g$ is even.
18. Prove the Difference Law: if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}[f(x) - g(x)] = L - M$. (Hint: $f - g = f + (-1)g$.)
19. Prove that if f is odd and the domain of f is symmetric about 0, then $f(0) = 0$ (if $f(0)$ is defined).
20. Use the Squeeze Theorem to prove that $\lim_{x\to 0} \sqrt{x}\sin(1/x) = 0$ (for $x > 0$).

Level 4 — Challenge

21. Evaluate $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$. *(Hint: use the known fact that $\sin x \approx x - x^3/6$ for small x; treat this as a challenge and verify your answer.)*
22. Show that $\lim\limits_{x\to 0} \sin(1/x)$ does not exist by constructing two sequences $x_n \to 0$ along which $\sin(1/x_n)$ has different limits.
23. Evaluate $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x} - x\right)$. *(Hint: multiply by $\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$.)*
24. A function f satisfies $|f(x)| \leq x^2$ for all x. Prove that $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} \frac{f(x)}{x} = 0$.
25. Let $f(x) = \lfloor x \rfloor$ (the greatest integer function). Find $\lim_{x\to n} f(x)$ for integer n, or explain why it does not exist.

Chapter 2

Continuity and Differentiation

Module II · 15 Hours · §1.5, §2.1, §2.2, §2.3, §2.5, §2.6 of Thomas & Finney (9th Ed.)

What you will learn in this chapter.

- What it means for a function to be continuous at a point or on an interval.
- The Intermediate Value Theorem and its applications.
- The derivative as a limit of a difference quotient, and its geometric meaning.
- The differentiation rules: constant, power, sum, product, quotient, and chain.
- How to differentiate implicitly defined functions.
- The power rule for rational exponents.

2.1 Continuity at a Point

Definition

A function f is **continuous at** $x = a$ if all three conditions hold:

1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

If any condition fails, f has a **discontinuity** at a.

Intuition

Continuity means the graph of f can be drawn through $(a, f(a))$ without lifting the pen. The three conditions prevent three types of failure: (1) the graph has a hole at a (function undefined); (2) the graph has a break (limit does not exist); (3) the graph has a filled-in misplaced dot (limit exists but $\neq f(a)$).

Example 2.1 — Checking continuity at a point

Is $f(x) = \dfrac{x^2 - 1}{x - 1}$ continuous at $x = 1$?

Solution.

1. $f(1)$: denominator is 0, so $f(1)$ is **undefined**. Condition (1) fails.

f has a discontinuity at $x = 1$.
However, $\lim_{x\to1} f(x) = \lim_{x\to1}(x+1) = 2$ (after cancelling $x-1$). This is a *removable* discontinuity: define $f(1) = 2$ to make f continuous.

Example 2.2 — A function continuous at $x = 0$

Is $g(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ continuous at $x = 0$?

Solution.

1. $g(0) = 0$. ✓
2. $\lim_{x\to0} x\sin(1/x) = 0$ by the Squeeze Theorem (since $|x\sin(1/x)| \leq |x| \to 0$). ✓
3. $\lim_{x\to0} g(x) = 0 = g(0)$. ✓

g is continuous at $x = 0$.

2.1.1 Types of Discontinuity

Definition

- **Removable discontinuity** at a: the limit exists but either $f(a)$ is undefined or $f(a) \neq \lim f$. Redefining $f(a) = \lim_{x\to a} f(x)$ removes the discontinuity.
- **Jump discontinuity** at a: the one-sided limits exist but are unequal.
- **Infinite discontinuity** at a: one or both one-sided limits are $\pm\infty$.
- **Oscillatory discontinuity** at a: the limit does not exist because the function oscillates wildly near a.

Example 2.3 — Removable discontinuity (Oct 2024 Q17)

Show that $f(x) = \dfrac{x^2+x-6}{x^2-4}$ has a removable discontinuity at $x = 2$ and extend f to be continuous there.

Solution. Factor: $x^2+x-6 = (x+3)(x-2)$ and $x^2-4 = (x-2)(x+2)$.

$$f(x) = \frac{(x+3)(x-2)}{(x-2)(x+2)} = \frac{x+3}{x+2} \quad (x \neq 2).$$

$\lim_{x\to2} f(x) = \dfrac{5}{4}$.
Since the limit exists but $f(2)$ is undefined, this is a **removable** discontinuity. Define $f(2) = 5/4$ to make f continuous at $x = 2$.

Example 2.4 — Jump discontinuity

$f(x) = \begin{cases} x+2 & x < 0 \\ x-1 & x \geq 0 \end{cases}$

$\lim_{x\to0^-} f(x) = 2$, $\lim_{x\to0^+} f(x) = -1$. Different: **jump discontinuity** at

$x = 0$.

Example 2.5 — Infinite discontinuity

$f(x) = 1/x$ at $x = 0$: $\lim_{x\to 0^+}(1/x) = +\infty$, $\lim_{x\to 0^-}(1/x) = -\infty$. **Infinite discontinuity.**

Example 2.6 — Oscillatory discontinuity

$f(x) = \sin(1/x)$ near $x = 0$: oscillates between -1 and 1 infinitely often; no one-sided limits exist. **Oscillatory (essential) discontinuity.**

2.2 Continuity on an Interval; Algebraic Properties

Theorem 2.1 (Algebra of Continuous Functions)**.** *If f and g are continuous at $x = a$, then $f + g$, $f - g$, fg, and f/g (provided $g(a) \neq 0$) are also continuous at a. Every polynomial is continuous everywhere. Every rational function is continuous wherever its denominator is nonzero.*

Proof for $f + g$

Since f and g are continuous at a: $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. By the Sum Law for limits: $\lim_{x\to a}[f(x) + g(x)] = f(a) + g(a) = (f + g)(a)$. So $f + g$ is continuous at a. The proofs for $f - g$ and fg are identical using the corresponding limit laws. □

2.3 The Intermediate Value Theorem

Theorem 2.2 (Intermediate Value Theorem (IVT))**.** *If f is **continuous** on $[a, b]$ and k is any value strictly between $f(a)$ and $f(b)$, then there exists at least one $c \in (a, b)$ with $f(c) = k$.*

Intuition

A continuous function cannot jump from $f(a)$ to $f(b)$ without passing through every intermediate value. Think of driving from city A to city B: you must pass through every city in between.

Example 2.7 — Existence of a root via IVT

Show that $f(x) = x^3 - x - 1$ has a root in $(1, 2)$.

Solution. f is a polynomial, hence continuous on $[1, 2]$. $f(1) = 1-1-1 = -1 < 0$. $f(2) = 8 - 2 - 1 = 5 > 0$.
Since $f(1) < 0 < f(2)$, IVT guarantees a $c \in (1, 2)$ with $f(c) = 0$.

Example 2.8 — Uniqueness using monotonicity + IVT

Show that $g(x) = x^3 + x + 1$ has exactly one real root.

Solution. *Existence:* $g(-1) = -1 - 1 + 1 = -1 < 0$ and $g(0) = 1 > 0$. By IVT, there is a root in $(-1, 0)$.
Uniqueness: $g'(x) = 3x^2 + 1 \geq 1 > 0$ for all x. So g is strictly increasing. A strictly monotone function can cross 0 at most once. Therefore the root is unique.

Example 2.9 — Every odd-degree polynomial has a real root

Prove that any polynomial $p(x)$ of odd degree has at least one real root.

Proof outline. A polynomial of odd degree n with positive leading coefficient satisfies: $p(x) \to +\infty$ as $x \to +\infty$ and $p(x) \to -\infty$ as $x \to -\infty$.
For large enough M: $p(-M) < 0 < p(M)$. Since p is a polynomial (continuous), IVT gives a root in $(-M, M)$. $\square$

2.4 The Derivative as a Limit

Definition

The **derivative** of f at $x = a$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists. If it exists, f is **differentiable** at a.

Intuition

The difference quotient $[f(a+h) - f(a)]/h$ is the slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$. As $h \to 0$, the secant line approaches the *tangent line* at $(a, f(a))$. The derivative $f'(a)$ is the slope of this tangent line.

Example 2.10 — Derivative of x^2 from the definition

Find $f'(x)$ for $f(x) = x^2$.

Solution.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) = 2x. \end{aligned}$$

Example 2.11 — Derivative of $1/x$

Find $f'(x)$ for $f(x) = 1/x$ ($x \neq 0$).

Solution.

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}.$$

Example 2.12 — Derivative of $\sqrt{x}$

Find $f'(x)$ for $f(x) = \sqrt{x}$ $(x > 0)$.

Solution.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Example 2.13 — Derivative of a constant

If $f(x) = c$ (a constant), then $f'(x) = \lim_{h\to 0} \frac{c-c}{h} = 0$.
This makes sense: a horizontal graph has slope 0 everywhere.

2.4.1 Tangent and Normal Lines

Definition

At the point $(a, f(a))$:

- The **tangent line** has slope $f'(a)$ and equation $y - f(a) = f'(a)(x - a)$.
- The **normal line** is perpendicular to the tangent; its slope is $-1/f'(a)$ (when $f'(a) \neq 0$).

Example 2.14 — Tangent line to a parabola

Find the equation of the tangent to $y = x^2 + 1$ at $x = 2$.

Solution. $f(2) = 5$, $f'(x) = 2x$, so $f'(2) = 4$. Tangent: $y - 5 = 4(x - 2)$, i.e. $y = 4x - 3$.

Example 2.15 — Normal to an implicit curve (Oct 2024 Q6)

Find the equation of the normal to $x^2 - xy + y^2 = 7$ at $(-1, 2)$.

Solution. Differentiate implicitly:

$$2x - (y + xy') + 2yy' = 0 \implies y'(2y - x) = y - 2x \implies y' = \frac{y - 2x}{2y - x}.$$

At $(-1, 2)$: $y' = \dfrac{2 - 2(-1)}{2(2) - (-1)} = \dfrac{4}{5}$.
Tangent slope $= 4/5$, so normal slope $= -5/4$. Normal line: $y - 2 = -\frac{5}{4}(x + 1)$, i.e. $4y - 8 = -5x - 5$, so $5x + 4y = 3$.

2.5 Differentiability and Continuity

Theorem 2.3. *If f is differentiable at a, then f is continuous at a.*

Proof

We want to show $\lim_{x\to a} f(x) = f(a)$. Write $f(x) - f(a) = \dfrac{f(x) - f(a)}{x - a} \cdot (x - a)$. As $x \to a$: the first factor $\to f'(a)$ (exists since f is differentiable at a) and the second factor $(x - a) \to 0$. So $\lim_{x\to a}[f(x) - f(a)] = f'(a) \cdot 0 = 0$, which means $\lim_{x\to a} f(x) = f(a)$. □

Warning

The converse is FALSE. A function can be continuous at a point without being differentiable there. The standard example is $f(x) = |x|$ at $x = 0$.

Example 2.16 — $|x|$ is continuous but not differentiable at 0

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Solution. The right-hand derivative: $\lim_{h\to 0^+} \dfrac{|h| - 0}{h} = \lim_{h\to 0^+} \dfrac{h}{h} = 1$.
The left-hand derivative: $\lim_{h\to 0^-} \dfrac{|h|}{h} = \lim_{h\to 0^-} \dfrac{-h}{h} = -1$.
The left and right limits differ, so $f'(0)$ does not exist. Yet f is clearly continuous at 0 (it equals 0 there and nearby).

Example 2.17 — Finding k for differentiability at a join

For what value of k is $f(x) = \begin{cases} kx^2 & x \leq 1 \\ 2x - 1 & x > 1 \end{cases}$ differentiable at $x = 1$?

Solution. *Continuity* requires $k(1)^2 = 2(1) - 1$, so $k = 1$.
Differentiability (with $k = 1$): left derivative $= 2k(1) = 2$; right derivative $= 2$. Equal. ✓
So $k = 1$.

2.6 The Basic Differentiation Rules

2.6.1 Constant, Power, and Sum Rules

Theorem 2.4.

$$\begin{aligned}
&\textit{(Constant)} && \frac{d}{dx}c = 0 \\
&\textit{(Power)} && \frac{d}{dx}x^n = nx^{n-1} \quad (n \textit{ any positive integer}) \\
&\textit{(Constant Multiple)} && \frac{d}{dx}[cf(x)] = cf'(x) \\
&\textit{(Sum/Difference)} && \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)
\end{aligned}$$

Proof of the Power Rule (n a positive integer)

Use the Binomial Theorem: $(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n$.

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1}.$$

As $h \to 0$, all terms with h vanish, leaving nx^{n-1}. □

Example 2.18 — Differentiating polynomials

Differentiate $f(x) = 5x^4 - 3x^3 + 7x - 2$.

Solution. $f'(x) = 5(4x^3) - 3(3x^2) + 7(1) - 0 = 20x^3 - 9x^2 + 7$.

Example 2.19 — Higher powers

Differentiate $g(x) = x^{10} - 3x^5 + 7x$.
$g'(x) = 10x^9 - 15x^4 + 7$.

2.6.2 The Product Rule

Theorem 2.5 (Product Rule). *If f and g are differentiable at x, then $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.*

Proof of the Product Rule

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to0}\left[\frac{f(x+h)-f(x)}{h}\cdot g(x+h)\right] + \lim_{h\to0}\left[f(x)\cdot\frac{g(x+h)-g(x)}{h}\right] \\
&= f'(x)\cdot g(x) + f(x)\cdot g'(x).
\end{aligned}$$

(We used: g is differentiable, hence continuous, so $g(x+h) \to g(x)$ as $h \to 0$.) $\square$

Example 2.20 — Product rule: two factors

Differentiate $f(x) = (x^2+1)(2x-3)$.

Solution. Let $u = x^2+1$, $v = 2x-3$. Then $u' = 2x$, $v' = 2$.

$$f'(x) = u'v + uv' = 2x(2x-3) + (x^2+1)(2) = 4x^2 - 6x + 2x^2 + 2 = 6x^2 - 6x + 2.$$

Check by expanding: $f(x) = 2x^3 - 3x^2 + 2x - 3$, so $f'(x) = 6x^2 - 6x + 2$. ✓

Example 2.21 — Product rule: three factors

Differentiate $h(x) = x \cdot (x-1) \cdot (x+2)$.

Solution. Write as $h = x \cdot [(x-1)(x+2)]$. Let $u = x$ and $v = (x-1)(x+2)$. $v' = (1)(x+2) + (x-1)(1) = 2x+1$. $h'(x) = 1 \cdot (x-1)(x+2) + x(2x+1) = (x^2+x-2) + 2x^2 + x = 3x^2 + 2x - 2$.
Alternatively: expand first, $h(x) = x^3 + x^2 - 2x$, giving $h'(x) = 3x^2 + 2x - 2$. ✓

Example 2.22 — Finding horizontal tangents

Find all x where the tangent to $y = (x^2-2)(x^2+3x)$ is horizontal.

Solution. $y' = 2x(x^2+3x) + (x^2-2)(2x+3) = 2x^3 + 6x^2 + 2x^3 + 3x^2 - 4x - 6 = 4x^3 + 9x^2 - 4x - 6$. Horizontal tangent: $y' = 0$. (This cubic can be approximated numerically or by the rational root theorem.) $4x^3 + 9x^2 - 4x - 6 = 0$; checking $x = 1/2$: $4(1/8) + 9(1/4) - 2 - 6 = 0.5 + 2.25 - 8 \neq 0$... One can verify the roots numerically.

2.6.3 The Quotient Rule

Theorem 2.6 (Quotient Rule)**.** *If f and g are differentiable at x and $g(x) \neq 0$:*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof via the Product Rule

Write $f = (f/g) \cdot g$. By the product rule: $f' = (f/g)'g + (f/g)g'$. Solving for $(f/g)'$: $(f/g)' = \frac{f' - (f/g)g'}{g} = \frac{f'g - fg'}{g^2}$. $\square$

Example 2.23 — Quotient rule

Differentiate $f(x) = \dfrac{x^2+1}{x-1}$.

Solution. $u = x^2 + 1$, $v = x - 1$. $u' = 2x$, $v' = 1$.

$$f'(x) = \frac{2x(x-1) - (x^2+1)(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}.$$

Example 2.24 — Differentiate $1/g(x)$

If $g(x) = 2x + 3$, find $\dfrac{d}{dx}\left[\dfrac{1}{g(x)}\right]$.

Solution. Apply the Quotient Rule with $f = 1$, $f' = 0$:

$$\left(\frac{1}{g}\right)' = \frac{0 \cdot g - 1 \cdot g'}{g^2} = \frac{-g'}{g^2} = \frac{-2}{(2x+3)^2}.$$

2.7 Rates of Change

Example 2.25 — Velocity and acceleration

The position of a particle is $s(t) = t^3 - 6t^2 + 9t$ (in metres, t in seconds). Find its velocity and acceleration. When is it at rest?

Solution. $v(t) = s'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$. $a(t) = v'(t) = 6t - 12$. At rest: $v(t) = 0 \Rightarrow t = 1$ or $t = 3$. Acceleration at $t = 1$: $a(1) = -6$ m/s^2 (decelerating); at $t = 3$: $a(3) = 6$ m/s^2 (accelerating).

Example 2.26 — Marginal cost

The cost of producing x units is $C(x) = 0.01x^3 - 0.6x^2 + 13x + 700$. Find the marginal cost at $x = 25$.

Solution. $C'(x) = 0.03x^2 - 1.2x + 13$. $C'(25) = 0.03(625) - 1.2(25) + 13 = 18.75 - 30 + 13 = 1.75$ per unit.

2.8 The Chain Rule

Theorem 2.7 (Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then $(f \circ g)$ is differentiable at x and:*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation: if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

Proof of the Chain Rule

Let $u = g(x)$ and $\Delta u = g(x+h) - g(x)$. Define $\varepsilon(k) = \frac{f(u+k)-f(u)}{k} - f'(u)$ for $k \neq 0$ and $\varepsilon(0) = 0$. Then $f(u+k) - f(u) = [f'(u) + \varepsilon(k)] \cdot k$, and since f is

differentiable at u, $\varepsilon(k) \to 0$ as $k \to 0$.
Dividing $f(g(x+h)) - f(g(x))$ by h:

$$\frac{f(g(x+h)) - f(g(x))}{h} = [f'(g(x)) + \varepsilon(\Delta u)] \cdot \frac{\Delta u}{h}.$$

As $h \to 0$: $\Delta u \to 0$ (since g is differentiable, hence continuous), so $\varepsilon(\Delta u) \to 0$; and $\Delta u/h \to g'(x)$. Limit: $f'(g(x)) \cdot g'(x)$. □

Example 2.27 — Basic chain rule

Differentiate $f(x) = (2x+1)^5$.

Solution. Outer function: u^5 (derivative $5u^4$). Inner function: $u = 2x+1$ (derivative 2). $f'(x) = 5(2x+1)^4 \cdot 2 = 10(2x+1)^4$.

Example 2.28 — Chain rule with a large power

$g(x) = (x^2+3x-1)^{10}$.
$g'(x) = 10(x^2+3x-1)^9 \cdot (2x+3) = 10(2x+3)(x^2+3x-1)^9$.

Example 2.29 — Nested chain rule

$h(x) = [(x+1)^2+1]^3$.
Let $u = (x+1)^2+1$ and $w = (x+1)$: $h'(x) = 3u^2 \cdot u' = 3[(x+1)^2+1]^2 \cdot [2(x+1) \cdot 1] = 6(x+1)[(x+1)^2+1]^2$.

Example 2.30 — Chain rule combined with product rule

Differentiate $k(x) = x^2(x^3+1)^4$.

Solution. $k = u \cdot v$ where $u = x^2$, $v = (x^3+1)^4$. $u' = 2x$, $v' = 4(x^3+1)^3 \cdot 3x^2 = 12x^2(x^3+1)^3$. $k'(x) = 2x(x^3+1)^4 + x^2 \cdot 12x^2(x^3+1)^3 = 2x(x^3+1)^3[(x^3+1)+6x^3] = 2x(7x^3+1)(x^3+1)^3$.

2.9 Implicit Differentiation

Intuition

Sometimes y is defined implicitly by an equation like $x^2+y^2 = 25$. We cannot always solve for y explicitly, but we can still find dy/dx by differentiating both sides with respect to x, treating y as a function of x and applying the chain rule wherever y appears.

Example 2.31 — Implicit differentiation: circle

Find dy/dx for $x^2+y^2 = 25$.

Solution. Differentiate both sides with respect to x:

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

At $(3, 4)$: $dy/dx = -3/4$ (slope of tangent to the circle).

Example 2.32 — Implicit differentiation with $\sin$ (Oct 2024 Q5)

Find dy/dx if $2y = x^2 + \sin y$.

Solution. Differentiate with respect to x:

$$2\frac{dy}{dx} = 2x + \cos y \cdot \frac{dy}{dx}.$$

Collect dy/dx terms:

$$\frac{dy}{dx}(2 - \cos y) = 2x \implies \frac{dy}{dx} = \frac{2x}{2 - \cos y}.$$

Example 2.33 — Folium of Descartes

Find dy/dx for $x^3 + y^3 = 6xy$.

Solution.

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}.$$

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2 \implies \frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}.$$

2.9.1 Second Derivative by Implicit Differentiation

Example 2.34 — Second derivative from $x^2 + y^2 = r^2$

Find y'' for $x^2 + y^2 = r^2$.

Solution. From implicit diff: $y' = -x/y$. Differentiate again (quotient rule on $-x/y$):

$$y'' = -\frac{y \cdot 1 - x \cdot y'}{y^2} = -\frac{y - x(-x/y)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{r^2}{y^3}.$$

2.10 Power Rule for Rational Exponents

Theorem 2.8 (Power Rule for Rational Exponents)**.** *For any rational number $r = p/q$ (in lowest terms, $q > 0$):*

$$\frac{d}{dx}x^r = rx^{r-1}.$$

Derivation via implicit differentiation

Let $y = x^{p/q}$, so $y^q = x^p$. Differentiate both sides implicitly:

$$qy^{q-1}y' = px^{p-1} \implies y' = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}}.$$

Substitute $y = x^{p/q}$:

$$y' = \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p(q-1)/q}} = \frac{p}{q} \cdot x^{p-1-p+p/q} = \frac{p}{q} x^{p/q-1}.$$

□

Example 2.35 — Rational exponent derivatives

Differentiate: (a) $f(x) = x^{3/2}$ (b) $g(x) = x^{2/3}$ (c) $h(x) = x^{-1/3}$

Solution.

(a) $f'(x) = \frac{3}{2}x^{1/2} = \frac{3\sqrt{x}}{2}$.
(b) $g'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$.
(c) $h'(x) = -\frac{1}{3}x^{-4/3} = \frac{-1}{3x^{4/3}}$.

Example 2.36 — Chain rule with rational exponents

Differentiate $k(x) = (x^2 + 1)^{1/2}$.
$k'(x) = \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$.

2.11 Chapter Summary

Concept	Key Statement
Continuity at a	3 conditions: $f(a)$ defined, limit exists, limit $= f(a)$
Types of discontinuity	Removable, jump, infinite, oscillatory
IVT	f cts on $[a, b]$, k between $f(a)$ and $f(b) \Rightarrow \exists c \in (a, b)$: $f(c) = k$
Derivative	$f'(a) = \lim_{h\to 0}[f(a + h) - f(a)]/h$
Differentiable ⇒ Continuous	Proved via $\lim_{x\to a} f(x) = f(a)$
Power Rule	$\frac{d}{dx}x^n = nx^{n-1}$ for any rational n
Product Rule	$(fg)' = f'g + fg'$
Quotient Rule	$(f/g)' = (f'g - fg')/g^2$
Chain Rule	$(f \circ g)' = f'(g(x)) \cdot g'(x)$
Implicit diff	Differentiate both sides wrt x; collect dy/dx

Key Takeaway

The Chain Rule is the most powerful differentiation tool: every differentiation is, at heart, an application of the chain rule to build up from simpler functions.

2.12 Exercises

Level 1 — Recall and Understand

1. State the three conditions for f to be continuous at $x = a$.
2. Show that $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$ has a removable discontinuity at $x = 2$ and define $f(2)$ to remove it. [Oct 2024 Q17]
3. Use the definition of the derivative to find $f'(x)$ for $f(x) = 3x^2$.
4. Find $f'(x)$ using the definition for $f(x) = \dfrac{1}{2x + 1}$.
5. Differentiate $p(x) = 4x^5 - 3x^3 + 2x - 7$.

Level 2 — Apply

6. Differentiate $f(x) = (x^2 + 1)(x^3 - 2x)$ using the product rule.
7. Differentiate $g(x) = \dfrac{3x - 5}{x^2 + 4}$ using the quotient rule.
8. Find the equation of the tangent to $y = x^3 - 3x$ at $x = 2$.
9. Find all x where $y = x^3 - 3x^2 + 3$ has a horizontal tangent.
10. Use the Chain Rule to differentiate $f(x) = (x^3 - 2x + 1)^{10}$.
11. Differentiate $g(x) = \sqrt{x^2 + 5x}$.
12. A particle moves along a line with position $s(t) = t^3 - 6t^2 + 9t$. Find the times when the particle is at rest and when it changes direction.
13. Use implicit differentiation to find dy/dx for $x^2 + xy + y^2 = 1$.
14. Find dy/dx for $2y = x^2 + \sin y$. [Oct 2024 Q5]
15. Find the equation of the normal to $x^2 - xy + y^2 = 7$ at $(-1, 2)$. [Oct 2024 Q6]
16. Differentiate $f(x) = x^{5/3} - 2x^{2/3}$.

Level 3 — Prove

17. Prove that if f is differentiable at a, then f is continuous at a.
18. Show that $f(x) = |x|$ is not differentiable at $x = 0$, even though it is continuous there.
19. Prove the Quotient Rule from the Product Rule.
20. Use the Intermediate Value Theorem to show that $\cos x = x$ has at least one solution. *(Hint: Consider $f(x) = \cos x - x$.)*
21. Let $f(x) = x^n$ for a positive integer n. Prove $f'(x) = nx^{n-1}$ using the factoring identity:
$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

Level 4 — Challenge

22. For what value of k is $f(x) = \begin{cases} x^2 & x \le 2 \\ kx + 2 & x > 2 \end{cases}$ (a) continuous at 2? (b) differentiable at 2? Can both conditions be satisfied simultaneously?
23. Find y'' for the curve $x^3 + y^3 = 1$ by implicit differentiation. Simplify your answer as much as possible.

24. Prove the Chain Rule using the definition of the derivative (handle the case $g'(x) = 0$ carefully).
25. A ladder of length L leans against a wall. The foot slides away from the wall at rate v m/s. Express the rate at which the top slides down the wall in terms of the current position x of the foot.
26. Differentiate $f(x) = (g(x))^n$ where g is differentiable and n is a positive integer. Use this to derive the product rule from the identity $(f + g)^2 = f^2 + 2fg + g^2$.

Chapter 3

Extreme Values and the Mean Value Theorem

Module III · 11 Hours · §3.1, §3.2, §3.3 of Thomas & Finney (9th Ed.)

What you will learn in this chapter.

- The Extreme Value Theorem: why continuous functions on closed intervals always have a maximum and minimum.
- How to find absolute extrema using the Closed Interval Method.
- Rolle's Theorem and the Mean Value Theorem, with complete proofs.
- How the sign of f' determines where f is increasing or decreasing.
- The First Derivative Test for classifying local extrema.

3.1 Absolute and Local Extreme Values

Definition

Let f be defined on a domain D.

- $f(c)$ is the **absolute maximum** of f on D if $f(c) \geq f(x)$ for all $x \in D$.
- $f(c)$ is the **absolute minimum** of f on D if $f(c) \leq f(x)$ for all $x \in D$.
- $f(c)$ is a **local maximum** if $f(c) \geq f(x)$ for all x near c (in some open interval about c).
- $f(c)$ is a **local minimum** if $f(c) \leq f(x)$ for all x near c.

Together, maxima and minima are called **extreme values** or **extrema**.

Intuition

An absolute maximum is the highest point on the entire graph; a local maximum is the highest point in its neighbourhood. Every absolute maximum is also a local maximum (if it occurs in the interior of the domain), but not vice versa.

Theorem 3.1 (Extreme Value Theorem (EVT))**.** *If f is **continuous** on a **closed, bounded** interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum on $[a, b]$.*

Example 3.1 — Why all three hypotheses of EVT are needed

Each condition in EVT is essential.

1. **Continuity fails:** $f(x) = \begin{cases} 1/x & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$ on $[0, 1]$. Has no maximum (unbounded near 0).
2. **Interval not closed:** $f(x) = x$ on $(0, 1)$. Approaches but never reaches 1; no maximum.
3. **Interval not bounded:** $f(x) = x$ on $[0, \infty)$. No maximum.

3.2 Critical Points and the Closed Interval Method

Theorem 3.2. *If f has a local extremum at an interior point c and $f'(c)$ exists, then $f'(c) = 0$.*

Proof

Suppose f has a local maximum at c (the minimum case is similar). Then for small $h > 0$: $[f(c+h) - f(c)]/h \leq 0$, giving the right-hand limit ≤ 0. For small $h < 0$: $[f(c+h) - f(c)]/h \geq 0$, giving the left-hand limit ≥ 0. Since both limits equal $f'(c)$, we need $f'(c) \leq 0$ and $f'(c) \geq 0$, so $f'(c) = 0$. □

Definition

An **interior critical point** of f is a point c in the interior of the domain where either:

- $f'(c) = 0$ (a **stationary point**), or
- $f'(c)$ does not exist.

Every local extremum at an interior point is a critical point, but not every critical point is an extremum.

Example 3.2 — Critical points where $f' = 0$

Find the critical points of $f(x) = x^3 - 3x + 1$.

Solution. $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$. $f'(x) = 0 \Rightarrow x = 1$ or $x = -1$. $f(1) = 1 - 3 + 1 = -1$ and $f(-1) = -1 + 3 + 1 = 3$. Critical points: $(-1, 3)$ and $(1, -1)$.

Example 3.3 — Critical point where f' does not exist

Find the critical points of $f(x) = x^{2/3}$.

Solution. $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$.
$f'(x) \neq 0$ for any x, but $f'(0)$ does not exist (the denominator is 0). So $x = 0$ is a critical point. The graph has a *cusp* at the origin.

Example 3.4 — Critical points of a trig function

Find the critical points of $f(x) = \sin x$ on $[0, 2\pi]$.
$f'(x) = \cos x = 0 \Rightarrow x = \pi/2, 3\pi/2$. $f(\pi/2) = 1$ (local max) and $f(3\pi/2) = -1$ (local min).

3.2.1 The Closed Interval Method

Definition

To find the **absolute extrema** of a continuous function f on $[a, b]$:

1. Find all critical points of f in (a, b).
2. Evaluate f at each critical point and at the endpoints a and b.
3. The largest value found is the absolute maximum; the smallest is the absolute minimum.

Example 3.5 — Closed interval method on a half-open interval (Oct 2024 Q7)

Find the absolute extrema of $f(x) = x^2 - 3$ on $[-2, 3)$.

Solution. $f'(x) = 2x = 0 \Rightarrow x = 0$ (critical point in $(-2, 3)$).
Evaluate:

x	$f(x)$
-2	$4 - 3 = 1$
0	$0 - 3 = -3$

The right endpoint $x = 3$ is **not** included; $f(x) \to 6$ as $x \to 3^-$ but 6 is never attained.
Absolute minimum: $f(0) = -3$ (at $x = 0$). **Absolute maximum:** does not exist (the supremum 6 is not attained).
This illustrates why EVT requires a *closed* interval.

Example 3.6 — Closed interval method (closed interval)

Find the absolute extrema of $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$.

Solution. $f'(x) = 6x^2 - 6x - 12 = 6(x - 2)(x + 1)$. Critical points in $(-2, 3)$: $x = 2$ and $x = -1$.

x	$f(x)$
-2	$-16 - 12 + 24 + 1 = -3$
-1	$-2 - 3 + 12 + 1 = 8$
2	$16 - 12 - 24 + 1 = -19$
3	$54 - 27 - 36 + 1 = -8$

Absolute maximum: $f(-1) = 8$. Absolute minimum: $f(2) = -19$.

Example 3.7 — Function with cusp

Find the absolute extrema of $f(x) = x^{2/3}$ on $[-1, 8]$.

Solution. $f'(x) = \frac{2}{3}x^{-1/3}$; does not exist at $x = 0$. $f'(x) \neq 0$ for all $x \neq 0$. Critical point: $x = 0$.

x	$f(x)$
-1	$(-1)^{2/3} = 1$
0	0
8	$8^{2/3} = 4$

Absolute maximum: $f(8) = 4$. Absolute minimum: $f(0) = 0$.

3.3 Rolle's Theorem

Theorem 3.3 (Rolle's Theorem). *Let f be continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then there exists at least one $c \in (a, b)$ with $f'(c) = 0$.*

Intuition

If you start and end at the same height on a smooth curve, somewhere in between you must momentarily be going neither up nor down (instantaneous slope $= 0$).

Proof of Rolle's Theorem

Since f is continuous on $[a, b]$, the Extreme Value Theorem guarantees f attains both its absolute maximum M and minimum m on $[a, b]$.
Case 1: $M = m$**.** Then f is constant on $[a, b]$, so $f'(c) = 0$ for every $c \in (a, b)$.
Case 2: $M > m$**.** Since $f(a) = f(b)$, at least one of M or m is attained at an interior point $c \in (a, b)$. At this interior extremum, Theorem 3.2 guarantees $f'(c) = 0$. □

Example 3.8 — Verifying Rolle's Theorem

Verify Rolle's Theorem for $f(x) = x^2 - 2x$ on $[0, 2]$.

Solution. f is a polynomial: continuous on $[0, 2]$, differentiable on $(0, 2)$. $f(0) = 0$, $f(2) = 4 - 4 = 0$. Equal. ✓
$f'(x) = 2x - 2 = 0 \Rightarrow x = 1 \in (0, 2)$. ✓
Rolle's Theorem is confirmed: the tangent is horizontal at $x = 1$.

Example 3.9 — Rolle's Theorem with three roots

Verify Rolle's Theorem for $f(x) = x(x - 1)(x - 2)$ on $[0, 2]$.

Solution. $f(0) = 0 = f(2)$. ✓($f(2) = 2 \cdot 1 \cdot 0 = 0$.) $f'(x) = 3x^2 - 6x + 2$. $f'(x) = 0 \Rightarrow x = \frac{6 \pm \sqrt{36-24}}{6} = 1 \pm \frac{1}{\sqrt{3}}$. Both values lie in $(0, 2)$. ✓(There are two such c.)

Example 3.10 — Rolle's Theorem for $\sin x$ on $[0, \pi]$

$f(0) = 0 = f(\pi) = \sin \pi$. $f'(x) = \cos x = 0$ at $x = \pi/2 \in (0, \pi)$. ✓

3.4 The Mean Value Theorem

Theorem 3.4 (Mean Value Theorem (MVT))**.** *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one $c \in (a, b)$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Intuition

The average rate of change of f over $[a, b]$ is $[f(b)-f(a)]/(b-a)$. The MVT says the *instantaneous* rate of change equals the average rate at least once. Geometrically: the tangent line at some interior point is parallel to the secant through the endpoints.

Real-world interpretation: If you drive from city A to city B (200 km) in 2 hours, your average speed is 100 km/h. MVT guarantees that at some moment, your speedometer read exactly 100 km/h.

Proof of the Mean Value Theorem

Define the auxiliary function:

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right].$$

The second term is the equation of the secant line from $(a, f(a))$ to $(b, f(b))$.
g is continuous on $[a, b]$ and differentiable on (a, b) (same as f). $g(a) = f(a)-f(a) = 0$ and $g(b) = f(b) - f(b) = 0$.
By Rolle's Theorem (Theorem 3.3), there exists $c \in (a, b)$ with $g'(c) = 0$.

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Setting $g'(c) = 0$: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

Example 3.11 — Applying MVT to find c

Find c guaranteed by MVT for $f(x) = x^2$ on $[1, 3]$.

Solution. $\frac{f(3)-f(1)}{3-1} = \frac{9-1}{2} = 4$. Solve $f'(c) = 4$: $2c = 4 \Rightarrow c = 2 \in (1, 3)$. ✓

Example 3.12 — MVT for x^3 on $[0, 2]$

$f(x) = x^3$; $\frac{f(2)-f(0)}{2-0} = \frac{8}{2} = 4$. Solve $f'(c) = 3c^2 = 4$: $c = 2/\sqrt{3} \approx 1.155 \in (0, 2)$. ✓

Example 3.13 — MVT for $\sin x$ on $[0, \pi/2]$

$\frac{\sin(\pi/2)-\sin(0)}{\pi/2-0} = \frac{1}{\pi/2} = \frac{2}{\pi}$. Solve $\cos c = 2/\pi$: $c = \arccos(2/\pi) \approx 0.881 \in (0, \pi/2)$. ✓

3.5 Consequences of the Mean Value Theorem

Theorem 3.5 (Corollary 1: Zero Derivative Implies Constant). *If $f'(x) = 0$ for all x in an open interval I, then f is constant on I.*

Proof

Take any $a, b \in I$ with $a < b$. By MVT, there exists $c \in (a, b)$ with $f(b) - f(a) = f'(c)(b - a) = 0 \cdot (b - a) = 0$. So $f(b) = f(a)$. Since a and b were arbitrary, f is constant. □

Example 3.14 — Application of Corollary 1 (Oct 2024 Q8)

If $f'(x) = 0$ for all x, then $f(x) = C$ (constant).
Application: If two functions have the same derivative, they differ by a constant. E.g., if $F'(x) = G'(x)$ for all x, then $(F - G)' = 0$, so $F - G = C$, i.e. $F(x) = G(x) + C$.

Theorem 3.6 (Corollary 2: Sign of Derivative Determines Monotonicity). *Let f be continuous on $[a, b]$ and differentiable on (a, b).*

- *If $f'(x) > 0$ on (a, b), then f is **strictly increasing** on $[a, b]$.*
- *If $f'(x) < 0$ on (a, b), then f is **strictly decreasing** on $[a, b]$.*

Proof (increasing case)

Take $x_1 < x_2$ in $[a, b]$. By MVT, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some $c \in (x_1, x_2)$. Since $f'(c) > 0$ and $x_2 - x_1 > 0$: $f(x_2) - f(x_1) > 0$, i.e. $f(x_2) > f(x_1)$. So f is strictly increasing. □

Example 3.15 — MVT-based inequality: $\sin x < x$ for $x > 0$

Prove $\sin x < x$ for all $x > 0$.

Proof. Let $g(x) = x - \sin x$. $g(0) = 0$. $g'(x) = 1 - \cos x \geq 0$ for all x (with equality only at $x = 2k\pi$). By MVT, for $x > 0$: $g(x) - g(0) = g'(c) \cdot x \geq 0$, so $g(x) \geq 0$. Since $g(x) = 0$ only at isolated points, $g(x) > 0$ for $x > 0$, i.e. $\sin x < x$. □

3.6 Increasing and Decreasing Functions

Procedure for finding intervals of increase/decrease:

1. Compute $f'(x)$.
2. Find where $f'(x) = 0$ or is undefined.
3. These points divide the domain into intervals.

4. Test the sign of f' on each interval (pick a test point).
5. $f' > 0$: increasing; $f' < 0$: decreasing.

Example 3.16 — Always increasing function (Oct 2024 Q19a)

Find the intervals on which $f(x) = x^3 + 12x + 5$ is increasing or decreasing on $[-3, 3]$.

Solution. $f'(x) = 3x^2 + 12 = 3(x^2 + 4)$. Since $x^2 + 4 \geq 4 > 0$ for all x, we have $f'(x) > 0$ everywhere.
f is **strictly increasing** on $[-3, 3]$ with no local extrema.
Absolute minimum: $f(-3) = -27 - 36 + 5 = -58$. Absolute maximum: $f(3) = 27 + 36 + 5 = 68$.

Example 3.17 — Increase/decrease with sign changes

Find increasing/decreasing intervals for $f(x) = x^4 - 8x^2$.

Solution. $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2)$.
Critical points: $x = -2, 0, 2$.
Sign chart of $f'(x)$:

Interval	Test pt	$4x$	$(x-2)(x+2)$	f'
$(-\infty, -2)$	-3	$-$	$(-)(-) = +$	$-$
$(-2, 0)$	-1	$-$	$(-1-2)(-1+2) = -$	$+$
$(0, 2)$	1	$+$	$(1-2)(1+2) = -$	$-$
$(2, \infty)$	3	$+$	$(+)(+) = +$	$+$

Increasing on $(-2, 0)$ and $(2, \infty)$. Decreasing on $(-\infty, -2)$ and $(0, 2)$.

Example 3.18 — Unique zero via monotonicity + IVT (Oct 2024 Q16)

Show that $f(x) = x^4 + 3x + 1$ has exactly one zero in $[-2, -1]$.

Solution.
Existence (IVT): $f(-2) = 16 - 6 + 1 = 11 > 0$ and $f(-1) = 1 - 3 + 1 = -1 < 0$.
f is a polynomial (continuous). By IVT, $\exists c \in (-2, -1)$ with $f(c) = 0$.
Uniqueness (monotonicity): $f'(x) = 4x^3 + 3$. On $[-2, -1]$: $x^3 \in [-8, -1]$, so $4x^3 \in [-32, -4]$, giving $f'(x) = 4x^3 + 3 \in [-29, -1] < 0$. So f is **strictly decreasing** on $[-2, -1]$.
A strictly monotone function can cross zero at most once. Therefore there is **exactly one** zero in $[-2, -1]$.

3.7 The First Derivative Test

Theorem 3.7 (First Derivative Test)**.** *Let c be a critical point of f, and suppose f is continuous at c.*

- *If f' changes from* ***positive to negative*** *at c: f has a* ***local maximum*** *at c.*
- *If f' changes from* ***negative to positive*** *at c: f has a* ***local minimum*** *at c.*

- *If f' does not change sign at c: f has **no local extremum** at c.*

Proof (local minimum case)

Suppose $f' < 0$ just left of c and $f' > 0$ just right of c. For $x < c$ near c: f is decreasing (Corollary 2), so $f(x) > f(c)$. For $x > c$ near c: f is increasing, so $f(x) > f(c)$. Therefore $f(c) \le f(x)$ for all x near c: $f(c)$ is a local minimum. $\square$

Example 3.19 — Classifying critical points with FDT

Find and classify all critical points of $f(x) = x^3 - 3x + 1$.

Solution. $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$. Critical points: $x = \pm 1$.
Sign of f':

Interval	f' sign	Behavior
$x < -1$	$+$	increasing
$-1 < x < 1$	$-$	decreasing
$x > 1$	$+$	increasing

At $x = -1$: f' changes $+ \to -$: **local maximum**, $f(-1) = 3$. At $x = 1$: f' changes $- \to +$: **local minimum**, $f(1) = -1$.

Example 3.20 — Classifying critical points of $x^4 - 8x^2$

From Example 3.17: f' changes as $-, +, -, +$ across $x = -2, 0, 2$.
At $x = -2$: $- \to +$: **local minimum**, $f(-2) = 16 - 32 = -16$. At $x = 0$: $+ \to -$: **local maximum**, $f(0) = 0$. At $x = 2$: $- \to +$: **local minimum**, $f(2) = 16 - 32 = -16$.

Example 3.21 — Critical point that is not an extremum

$f(x) = x^3$; $f'(x) = 3x^2$; $f'(0) = 0$ but $f' \ge 0$ everywhere. f' does NOT change sign at $x = 0$. So $x = 0$ is a critical point (stationary point) but **not an extremum** — it is an inflection point.

3.8 Antiderivatives

Definition

F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

By Corollary 2, if F and G are both antiderivatives of f on I, then $(F - G)' = 0$, so $F(x) = G(x) + C$ for some constant C.

Example 3.22 — All antiderivatives of a power

The antiderivatives of $f(x) = 2x$ are all functions of the form $F(x) = x^2 + C$.

More generally: the antiderivatives of x^n (for $n \neq -1$) are $\dfrac{x^{n+1}}{n+1} + C$.

3.9 Chapter Summary

Concept	Key Statement
EVT	f cts on $[a, b] \Rightarrow f$ attains absolute max and min
Critical point	Interior c where $f'(c) = 0$ or $f'(c)$ DNE
Closed Interval Method	Evaluate f at critical pts + endpoints
Rolle's Theorem	$f(a) = f(b) \Rightarrow \exists c \in (a, b)$: $f'(c) = 0$ (complete proof)
MVT	$\exists c$: $f'(c) = [f(b) - f(a)]/(b - a)$ (proved via Rolle)
$f' = 0$ everywhere	f constant (Corollary 1)
$f' > 0$ on (a, b)	f increasing; $f' < 0 \Rightarrow$ decreasing
First Derivative Test	Sign change of f' at c determines local extremum type
Antiderivative	$F' = f$; all antiderivatives differ by a constant C

Key Takeaway

The Mean Value Theorem is the backbone of differential calculus: it is the reason the sign of f' controls where f increases, and it is the key step in the proofs of the First and Second Derivative Tests.

3.10 Exercises

Level 1 — Recall and Understand

1. Find all critical points of $f(x) = x^3 - 6x^2 + 9x$.
2. State the Extreme Value Theorem. Give one example where each hypothesis is essential.
3. Find the absolute extrema of $f(x) = x^2 - 3$ on $[-2, 3)$. [Oct 2024 Q7]
4. Verify that $f(x) = x^2 - 4x$ on $[0, 4]$ satisfies the hypotheses of Rolle's Theorem, and find c.
5. Give an example of a function on $[0, 1]$ that has no local maximum or minimum at $x = 0$. [Oct 2024 Q9]

Level 2 — Apply

6. Find the absolute maximum and minimum of $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$.
7. Find all intervals on which $f(x) = x^3 + 12x + 5$ is increasing or decreasing on $[-3, 3]$. Find all extreme values. [Oct 2024 Q19a]
8. Show that $f(x) = x^4 + 3x + 1$ has exactly one zero in $[-2, -1]$. [Oct 2024 Q16]
9. Find all local extrema of $f(x) = x^4 - 8x^2 + 5$ using the First Derivative Test.

10. Find c satisfying the MVT for $f(x) = x^3 - x$ on $[-1, 2]$.
11. A function f satisfies $f(1) = 3$ and $f'(x) \leq 2$ for all $x \in [1, 4]$. Using the MVT, find the largest possible value of $f(4)$.
12. Find the increasing/decreasing intervals of $f(x) = \dfrac{x^2}{x+1}$.
13. Show that $x^3 + x + 1 = 0$ has exactly one real root.
14. Find all local extrema of $f(x) = x^{2/3}(x - 4)$.

Level 3 — Prove

15. Prove that if $f'(x) = 0$ for all $x \in I$, then f is constant on I. [Oct 2024 Q8]
16. Prove that if $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ for some constant C.
17. Prove the First Derivative Test (local maximum case): if f' changes from $+$ to $-$ at c, then $f(c)$ is a local maximum.
18. Use the Mean Value Theorem to prove $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.
19. Let f be continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Use Rolle's Theorem to prove that f' has at least one zero in (a, b).

Level 4 — Challenge

20. Prove that a polynomial of degree n has at most $n - 1$ critical points, and hence at most n local extrema.
21. Let f be differentiable on $\mathbb{R}$ with $f(0) = 0$ and $|f'(x)| \leq M$ for all x. Prove that $|f(x)| \leq M|x|$ for all x.
22. Prove that the equation $x^5 + 5x + 1 = 0$ has exactly one real root.
23. If f is twice differentiable and $f(a) = f(b) = 0$ with $f(c) > 0$ for some $c \in (a, b)$, prove that $f''(d) < 0$ for some $d \in (a, b)$. *(Hint: Apply MVT twice.)*
24. Show that if f is continuous on $[a, b]$ and differentiable on (a, b) with $f'(x) \neq 0$ on (a, b), then f is **strictly monotone** on $[a, b]$.

Chapter 4

Graphing and Asymptotes

Module IV · 10 Hours · §3.4, §3.5 of Thomas & Finney (9th Ed.)

What you will learn in this chapter.

- How to use the second derivative to determine concavity and locate inflection points.
- The Second Derivative Test for classifying critical points.
- How to find horizontal, vertical, and oblique asymptotes.
- A systematic strategy for sketching the complete graph of a function.

4.1 Concavity and the Second Derivative

Definition

The graph of a differentiable function f is:

- **concave up** on an interval I if f' is **increasing** on I (equivalently, $f''(x) > 0$ on I).
- **concave down** on I if f' is **decreasing** on I (equivalently, $f''(x) < 0$ on I).

Intuition

Concave up: the curve "holds water" (like a bowl); the tangent line lies *below* the curve. Concave down: the curve is "upside-down bowl"; the tangent line lies *above* the curve. The second derivative f'' measures the rate of change of the slope f'. If the slope is increasing, the curve bends upward (concave up).

Example 4.1 — Determining concavity from f''

Determine the concavity of $f(x) = x^3 - 6x^2$.

Solution. $f'(x) = 3x^2 - 12x$. $f''(x) = 6x - 12 = 6(x - 2)$.
$f''(x) < 0$ for $x < 2$: **concave down** on $(-\infty, 2)$. $f''(x) > 0$ for $x > 2$: **concave up** on $(2, \infty)$.

Example 4.2 — Concavity of a polynomial

$f(x) = x^3$: $f''(x) = 6x$. Concave down for $x < 0$; concave up for $x > 0$.

Example 4.3 — Always concave up

$f(x) = x^2$: $f''(x) = 2 > 0$ everywhere. f is concave up on all of $\mathbb{R}$.

4.2 Inflection Points

Definition

A **point of inflection** (or **inflection point**) is a point $(c, f(c))$ on the graph where:

1. The curve has a tangent line (i.e. f is continuous at c).
2. The concavity **changes** at c (from concave up to concave down, or vice versa).

If $f''(c) = 0$ and f'' changes sign at c, then $(c, f(c))$ is an inflection point.

Warning

$f''(c) = 0$ is necessary but NOT sufficient for an inflection point. Check that f'' actually **changes sign** at c. Counterexample: $f(x) = x^4$, $f''(0) = 0$, but $f'' = 12x^2 \geq 0$ always — no inflection at $x = 0$.

Example 4.4 — Finding inflection points

Find the inflection points of $f(x) = x^3 - 3x$.

Solution. $f''(x) = 6x$. $f''(0) = 0$; for $x < 0$: $f'' < 0$ (concave down); for $x > 0$: $f'' > 0$ (concave up). f'' changes sign at $x = 0$: **inflection point** at $(0, 0)$.

Example 4.5 — Two inflection points

Find inflection points of $f(x) = x^4 - 6x^2$.

Solution. $f'(x) = 4x^3 - 12x$. $f''(x) = 12x^2 - 12 = 12(x^2 - 1) = 12(x-1)(x+1)$.
$f''(x) = 0$ at $x = \pm 1$.
Sign of f'': $+$ for $|x| > 1$, $-$ for $|x| < 1$.
At $x = -1$: f'' changes $+ \to -$: inflection at $(-1, f(-1)) = (-1, -5)$. At $x = 1$: f'' changes $- \to +$: inflection at $(1, -5)$.

Example 4.6 — No inflection despite $f'' = 0$

$f(x) = x^4$: $f''(x) = 12x^2 \geq 0$ for all x; $f''(0) = 0$ but no sign change. **No inflection point** at $x = 0$; the graph is concave up everywhere.

4.3 The Second Derivative Test

Theorem 4.1 (Second Derivative Test (SDT)). *Let $f'(c) = 0$ (so c is a critical point).*

- *If $f''(c) > 0$: f has a* ***local minimum*** *at c.*
- *If $f''(c) < 0$: f has a* ***local maximum*** *at c.*
- *If $f''(c) = 0$: the test is* ***inconclusive****; use the First Derivative Test.*

Proof (local minimum case)

Suppose $f'(c) = 0$ and $f''(c) > 0$. Then $f''(c) = \lim_{h\to 0} \dfrac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \dfrac{f'(c+h)}{h} > 0$.
For small $h > 0$: $f'(c+h)/h > 0$, so $f'(c+h) > 0$ (slope positive right of c). For small $h < 0$: $f'(c+h)/h > 0$ but $h < 0$, so $f'(c+h) < 0$ (slope negative left of c). By the First Derivative Test, $f(c)$ is a local minimum. $\square$

Example 4.7 — SDT applied to $x^3 - 3x$

Classify the critical points of $f(x) = x^3 - 3x$.

Solution. $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$. $f''(x) = 6x$.
$f''(-1) = -6 < 0$: **local maximum** at $x = -1$; $f(-1) = 2$. $f''(1) = 6 > 0$: **local minimum** at $x = 1$; $f(1) = -2$.

Example 4.8 — SDT inconclusive for x^4

$f(x) = x^4$: $f'(0) = 0$, $f''(0) = 0$: SDT is inconclusive.
Use FDT: $f'(x) = 4x^3$; negative for $x < 0$, positive for $x > 0$. Sign changes $- \to +$: **local (and global) minimum** at $x = 0$.

Example 4.9 — SDT applied to a cubic with two critical points

$f(x) = x^3 - 6x^2 + 9x + 1$.
$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. Critical points: $x = 1, 3$. $f''(x) = 6x - 12$.
$f''(1) = -6 < 0$: **local maximum** at $x = 1$; $f(1) = 1-6+9+1 = 5$. $f''(3) = 6 > 0$: **local minimum** at $x = 3$; $f(3) = 27 - 54 + 27 + 1 = 1$.

4.4 Limits at Infinity and Horizontal Asymptotes

Definition

The line $y = L$ is a **horizontal asymptote** of f if:

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

A function can have at most two horizontal asymptotes (one as $x \to +\infty$ and one as $x \to -\infty$, possibly different).

Example 4.10 — Horizontal asymptote of a rational function

Find the horizontal asymptote of $f(x) = \dfrac{2x+3}{x-1}$.

Solution. Divide by x: $\dfrac{2+3/x}{1-1/x} \to \dfrac{2}{1} = 2$ as $x \to \pm\infty$. Horizontal asymptote: $y = 2$.

Example 4.11 — Rational function with same-degree numerator and denominator

$\lim\limits_{x\to\infty} \dfrac{3x^2-2}{2x^2+5x-3}$.
Divide by x^2: $\dfrac{3-2/x^2}{2+5/x-3/x^2} \to \dfrac{3}{2}$. Horizontal asymptote: $y = 3/2$.

Example 4.12 — Two different horizontal asymptotes

$f(x) = \dfrac{x}{\sqrt{x^2+1}}$.
As $x \to +\infty$: $f(x) \to 1$ (shown in Chapter 1). As $x \to -\infty$: $f(x) \to -1$.
$y = 1$ (as $x \to +\infty$) and $y = -1$ (as $x \to -\infty$) are two distinct horizontal asymptotes.

Example 4.13 — The Squeeze gives a horizontal asymptote (Oct 2024 Q17)

Show that $y = 2 + \dfrac{\sin x}{x}$ has horizontal asymptote $y = 2$.

Solution. $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$ (by Squeeze Theorem: $|\sin x| \le 1$, so $|\sin x/x| \le 1/|x| \to 0$).
Therefore $\lim\limits_{x\to\infty} \left(2 + \dfrac{\sin x}{x}\right) = 2 + 0 = 2$.
Horizontal asymptote: $y = 2$.
Note: the function oscillates across the asymptote infinitely many times, since $\sin x$ changes sign infinitely often. This is allowed — a HA is a limit, not a barrier.

4.5 Vertical Asymptotes

Definition

The line $x = a$ is a **vertical asymptote** of f if at least one of the following holds:

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

Example 4.14 — Vertical asymptote with simplification needed

Find the vertical asymptotes of $f(x) = \dfrac{x+1}{x^2-1}$.

Solution. $x^2 - 1 = (x-1)(x+1)$. For $x \neq -1$: $f(x) = \dfrac{x+1}{(x-1)(x+1)} = \dfrac{1}{x-1}$.
At $x = 1$: $\lim_{x\to 1} \frac{1}{x-1} = \pm\infty \to$ **vertical asymptote** at $x = 1$. At $x = -1$: the factor $(x+1)$ cancels $\to$ **removable** discontinuity (not a VA).

Example 4.15 — Sign analysis near a vertical asymptote

Determine the signs of $\displaystyle\lim_{x\to 2^\pm} \frac{x-3}{x-2}$.

Solution. As $x \to 2^+$: numerator $\to -1 < 0$; denominator $\to 0^+$. So limit $= -\infty$.
As $x \to 2^-$: numerator $\to -1 < 0$; denominator $\to 0^-$. So limit $= +\infty$.
VA at $x = 2$.

4.6 Oblique Asymptotes

Definition

A line $y = mx + b$ (with $m \neq 0$) is an **oblique asymptote** of f if:

$$\lim_{x\to\pm\infty} [f(x) - (mx + b)] = 0.$$

An oblique asymptote arises when $f(x)$ is a rational function where the degree of the numerator is exactly 1 more than the degree of the denominator.

Example 4.16 — Oblique asymptote via long division

Find the oblique asymptote of $f(x) = \dfrac{x^2+1}{x}$.

Solution. Perform the division: $\dfrac{x^2+1}{x} = x + \dfrac{1}{x}$.
As $x \to \pm\infty$: $\dfrac{1}{x} \to 0$, so $f(x) \to x$. Oblique asymptote: $y = x$. Also: vertical asymptote at $x = 0$.

Example 4.17 — Oblique asymptote for $f(x) = (x^3+1)/(x^2)$

Divide: $\dfrac{x^3+1}{x^2} = x + \dfrac{1}{x^2}$. As $x \to \pm\infty$: $1/x^2 \to 0$. Oblique (actually linear) asymptote: $y = x$. Note: since the numerator is degree 3 and denominator is degree 2, degree difference is 1, so yes, oblique asymptote.

Example 4.18 — General method for oblique asymptote

Find the asymptotes of $f(x) = \dfrac{x^2 + x + 1}{x - 1}$.

Solution. Long division: $x^2 + x + 1 = (x - 1)(x + 2) + 3$, so $f(x) = x + 2 + \dfrac{3}{x - 1}$. As $x \to \pm\infty$: $3/(x - 1) \to 0$. Oblique asymptote: $y = x + 2$. Vertical asymptote: $x = 1$.

4.7 Complete Curve Sketching

Systematic Curve Sketching Strategy

1. **Domain** — Where is f defined?
2. **Intercepts** — y-intercept ($x = 0$); x-intercepts ($f(x) = 0$).
3. **Symmetry** — Is f even, odd, or periodic?
4. **Asymptotes** — Vertical (where $f \to \pm\infty$); Horizontal ($x \to \pm\infty$); Oblique.
5. **First derivative** — Find f'; determine critical points; sign chart of f'.
6. **Increasing/Decreasing and local extrema** — From the sign chart.
7. **Second derivative** — Find f''; find inflection points; sign chart of f''.
8. **Concavity** — From the sign chart of f''.
9. **Sketch** — Plot intercepts, asymptotes, critical points, inflection points; draw curves.

Example 4.19 — Complete analysis of $(x^3 + 1)/x$ (Oct 2024 Q20)

Sketch $f(x) = \dfrac{x^3 + 1}{x}$.

Solution.

1. **Domain:** $x \neq 0$, so $(-\infty, 0) \cup (0, \infty)$.
2. **Intercepts:** x-intercepts: $x^3 + 1 = 0 \Rightarrow x = -1$. Point: $(-1, 0)$. y-intercept: $f(0)$ undefined.
3. **Symmetry:** $f(-x) = \dfrac{-x^3 + 1}{-x} = \dfrac{x^3 - 1}{x}$, which is neither $f(x)$ nor $-f(x)$. No symmetry.
4. **Asymptotes:** Vertical: $x = 0$ (denominator zero). $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^3+1}{x} = +\infty$ (numerator $\to 1 > 0$, denominator $\to 0^+$). $\lim_{x\to 0^-} f(x) = -\infty$. Oblique: $f(x) = x^2 + \dfrac{1}{x}$; as $x \to \pm\infty$, $1/x \to 0$, so $f(x) \approx x^2$. (Parabolic asymptote, not a straight line. No linear oblique asymptote here since the degree difference is $3 - 1 = 2$.)
5. **First derivative:** $f(x) = x^2 + x^{-1}$, so $f'(x) = 2x - x^{-2} = \dfrac{2x^3 - 1}{x^2}$. $f'(x) = 0 \Rightarrow 2x^3 = 1 \Rightarrow x = \dfrac{1}{\sqrt[3]{2}} = 2^{-1/3} \approx 0.794$.
   Sign of f':

x interval	$2x^3 - 1$	f' sign
$(-\infty, 0)$	< 0 (since $x < 0$)	$-$ (decreasing)
$(0, 2^{-1/3})$	< 0	$-$ (decreasing)
$(2^{-1/3}, \infty)$	> 0	$+$ (increasing)

Local minimum at $x = 2^{-1/3}$: $f(2^{-1/3}) = (2^{-1/3})^2 + 2^{1/3} = 2^{-2/3} + 2^{1/3} = \frac{3}{2^{2/3}} = \frac{3 \cdot 2^{1/3}}{2}$.

6. **Second derivative:** $f''(x) = 2 + 2x^{-3} = \dfrac{2x^3+2}{x^3} = \dfrac{2(x^3+1)}{x^3}$. $f''(x) = 0 \Rightarrow x^3 = -1 \Rightarrow x = -1$. f'' changes sign at $x = -1$: **inflection point** at $(-1, 0)$. Concave up: $x^3 + 1 > 0$ and $x^3 > 0 \to x > 0$; or $x^3 + 1 < 0$ and $x^3 < 0 \to x < -1$. Concave down: $-1 < x < 0$.

The curve passes through $(-1, 0)$, has a VA at $x = 0$, a local min near $(0.794, 1.89)$, and an inflection at $(-1, 0)$.

Example 4.20 — Sketching $f(x) = x^2/(x-1)$

Analysis:

- Domain: $x \neq 1$.
- y-intercept: $f(0) = 0$; x-intercept: $x = 0$.
- VA: $x = 1$; $\lim_{x\to1^+} = +\infty$, $\lim_{x\to1^-} = -\infty$.
- Oblique: $x^2/(x-1) = x+1+1/(x-1)$; so $y = x+1$ is an oblique asymptote.
- $f'(x) = \dfrac{2x(x-1) - x^2}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$.
- Critical points: $x = 0$ and $x = 2$.
- $f' < 0$ on $(0, 1)$ and $(0$, before split): sign chart shows local max at $x = 0$ ($f(0) = 0$) and local min at $x = 2$ ($f(2) = 4$).
- $f''(x) = \dfrac{2}{(x-1)^3}$; concave down for $x < 1$, concave up for $x > 1$; no inflection (f not defined at $x = 1$).

Example 4.21 — Constructing a graph from limit/derivative information (Oct 2024 Q18)

Given:

- $f(0) = 1$, $f'(0) = 0$ (horizontal tangent at $x = 0$)
- $\lim_{x\to\pm\infty} f(x) = 2$ (horizontal asymptote $y = 2$)
- $f'(x) > 0$ for $x < 0$; $f'(x) < 0$ for $x > 0$
- $f''(x) < 0$ for all x (concave down everywhere)

Sketch a possible graph of f.

Interpretation: f increases to a local max at $(0, 1)$, then decreases toward $y = 2$ from below. But wait: $f(0) = 1 < 2$ and the function approaches $y = 2$ from below. So f increases up to 1 at $x = 0$, but the HA is $y = 2 > 1$?
Let's re-read: f increases to $x = 0$ (max at $x = 0$), then decreases. Since $f(0) = 1$ is the maximum and $f \to 2$ at infinity... but $2 > 1$ means f would have to increase again!

This contradiction means the description as given is inconsistent. *A correct version*: $f(0) = 3$, $\lim f = 2$, $f' > 0$ left of 0, $f' < 0$ right of 0. Then: f rises from $y = 2$, peaks at $f(0) = 3$, falls back to $y = 2$. The graph is like an arch over the HA.

Example 4.22 — Full sketch of $f(x) = x^3 - 3x^2$

- Domain: $\mathbb{R}$. Intercepts: $f(0) = 0$; $f(x) = 0 \Rightarrow x^2(x-3) = 0 \Rightarrow x = 0, 3$.
- $f'(x) = 3x^2 - 6x = 3x(x-2)$. Critical pts: $x = 0$ (local max, $f(0) = 0$) and $x = 2$ (local min, $f(2) = -4$).
- $f''(x) = 6x - 6 = 6(x-1)$. Inflection at $x = 1$: $f(1) = -2$.
- Concave down on $(-\infty, 1)$; concave up on $(1, \infty)$.
- No asymptotes (polynomial).
- $f \to -\infty$ as $x \to -\infty$; $f \to +\infty$ as $x \to +\infty$.

4.8 Chapter Summary

Concept	Key Statement
Concave up/down	$f'' > 0$: concave up; $f'' < 0$: concave down
Inflection point	f'' changes sign; $f''(c) = 0$ is necessary but not sufficient
Second Derivative Test	$f'(c) = 0$, $f''(c) > 0$: local min; $f''(c) < 0$: local max; $f''(c) = 0$: inconclusive
Horizontal asymptote	$y = L$ if $\lim_{x\to\pm\infty} f(x) = L$
Vertical asymptote	$x = a$ if $f \to \pm\infty$ as $x \to a^{\pm}$
Oblique asymptote	$y = mx + b$ if $f(x) - (mx + b) \to 0$; found by long division
Curve sketching	9-step strategy: domain, intercepts, symmetry, asymptotes, f', f'', sketch

Key Takeaway

The second derivative f'' tells you the shape of the graph ($f'' > 0$ = bowl, $f'' < 0$ = arch). Combined with the first derivative, you can read off the graph's complete story: where it rises, falls, bends, and what it approaches at infinity.

4.9 Exercises

Level 1 — Recall and Understand

1. Find the intervals of concavity and any inflection points for $f(x) = x^3 - 3x$.
2. Determine the concavity of $f(x) = x^4 - 6x^2$ and find all inflection points.
3. Use the Second Derivative Test to classify the critical points of $f(x) = x^3 - 6x^2 + 9x$.
4. Find the horizontal asymptote(s) of $f(x) = \dfrac{3x+5}{x-2}$.

5. Find all vertical asymptotes of $g(x) = \dfrac{x+2}{x^2-4}$.

Level 2 — Apply

6. Find and classify all critical points of $f(x) = x^4 - 8x^2 + 5$ using the Second Derivative Test. Find the inflection points. [Oct 2024 Q21b (part)]
7. Find the horizontal asymptote(s) of $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.
8. Show that $y = 2 + \dfrac{\sin x}{x}$ has horizontal asymptote $y = 2$. Explain whether the graph crosses the asymptote. [Oct 2024 Q17]
9. Find the oblique asymptote of $f(x) = \dfrac{x^2+3}{x-1}$.
10. Find the oblique asymptote of $f(x) = \dfrac{2x^2-x+3}{x+2}$.
11. Sketch the graph of $f(x) = x^3 - 3x^2 + 3$, showing all critical points, inflection points, and behavior as $x \to \pm\infty$.
12. Sketch the graph of $y = \dfrac{x^3+1}{x}$ using the 9-step strategy. [Oct 2024 Q20]
13. Sketch the graph of $f(x) = \dfrac{x^2}{x-1}$, including all asymptotes and key features.
14. Find the inflection points of $f(x) = xe^{-x^2}$. *(Use the product/chain rule; treat e^{-x^2} as a known function.)*
15. Given that $f(x) = (x-1)^2(x+2)$, find all critical points, classify them, find inflection points, and sketch the graph.

Level 3 — Prove

16. Prove the Second Derivative Test (local maximum case): if $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
17. Prove that a function f is concave up on (a, b) if and only if for all $x_1, x_2 \in (a, b)$ with $x_1 < x_2$ and any $t \in (0, 1)$:
$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2).$$
(This is the definition of a convex function; prove the "only if" direction using the MVT.)
18. Prove that if f has an inflection point at c and $f''(c)$ exists, then $f''(c) = 0$. *(Hint: Apply the argument of Theorem 3.2 to f'.)*
19. Prove that a rational function $p(x)/q(x)$ (in lowest terms) has a horizontal asymptote $y = 0$ if $\deg p < \deg q$, and $y = a_n/b_m$ (ratio of leading coefficients) if $\deg p = \deg q$.
20. Prove that if f is twice differentiable and $f''(x) > 0$ on (a, b), then the graph of f lies above every tangent line on (a, b).

Level 4 — Challenge

21. A function satisfies: $f(0) = 3$, $\lim_{x\to\pm\infty} f(x) = 2$, $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$, $f''(x) < 0$ for all x. Sketch a possible graph, labelling all key features.
22. Show that $f(x) = x^{2/3}(6-x)^{1/3}$ has inflection points and identify them. *(Compute $f'(x)$ using the product rule and rational exponent rule; simplify $f''(x)$ carefully.)*
23. Find a polynomial $f(x)$ of degree 4 with a local maximum at $x = 0$, local minima at $x = \pm 2$, inflection points at $x = \pm 1$, and $f(0) = 4$.
24. If f is differentiable on $\mathbb{R}$ and the graph of f lies below every one of its tangent lines, prove that f can have at most one zero.

25. The *logistic function* $f(x) = \dfrac{1}{1+e^{-x}}$ (where e^{-x} is a given function with derivative $-e^{-x}$) satisfies $f'(x) = f(x)(1 - f(x))$. Find the inflection point of f. What is the horizontal asymptote?

Chapter 5

Trigonometry, Optimization, and Linearization

Module V · 12 Hours · Open-Ended

Exam Tip

This chapter is assessed in the internal examination only (Module V). It does not appear in the 70-mark external examination, which covers Chapters 1–4 only. Study this material to deepen your understanding and for the internal 30 marks.

What you will learn in this chapter.

- Radian measure and the unit circle; all six trigonometric functions and their derivatives.
- The formal ε-δ definition of a limit, with complete proofs of basic limits.
- The fundamental limit $\lim_{x\to 0}(\sin x)/x = 1$ via a geometric squeeze argument.
- Optimization problems: setting up and solving max/min problems with constraints.
- Linearization: approximating $f(x)$ near $x = a$ by the tangent line $L(x) = f(a) + f'(a)(x - a)$.
- Differentials and error estimation.

5.1 Radian Measure and the Unit Circle

Definition

One **radian** is the angle subtended at the centre of a circle of radius r by an arc of length r. For a circle of radius r:

$$\text{Arc length} = r\theta \quad (\theta \text{ in radians}), \qquad \text{Sector area} = \tfrac{1}{2}r^2\theta.$$

Conversion: $180° = \pi$ radians.

Example 5.1 — Radian conversions

(a) Convert 135° to radians. (b) Convert $\frac{5\pi}{6}$ to degrees.

Solution. (a) $135° = 135 \times \frac{\pi}{180} = \frac{3\pi}{4}$ radians.
(b) $\frac{5\pi}{6} \times \frac{180°}{\pi} = \frac{5 \times 180°}{6} = 150°$.

Example 5.2 — Arc length and sector area

A circle has radius 5 cm. (a) Find the arc length subtended by an angle of $\frac{\pi}{3}$. (b) Find the area of the corresponding sector.

Solution. (a) $s = r\theta = 5 \cdot \frac{\pi}{3} = \frac{5\pi}{3}$ cm.
(b) $A = \frac{1}{2}r^2\theta = \frac{1}{2}(25) \cdot \frac{\pi}{3} = \frac{25\pi}{6}$ cm^2.

5.1.1 Standard Trigonometric Values

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef	0	undef

5.2 Trigonometric Identities

Definition

Fundamental identities:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \quad (= 1 + \sin^2/\cos^2 = 1/\cos^2) \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

Addition formulas:

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned}$$

Double angle:

$$\begin{aligned} \sin 2A &= 2\sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A \end{aligned}$$

Example 5.3 — Using addition formulas

Compute $\sin(75°) = \sin(45° + 30°)$.

Solution. $\sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$.

5.3 The Formal ε-δ Definition of Limit

Definition

We say $\lim_{x\to a} f(x) = L$ if: for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Intuition

ε is a challenge: "how close to L must $f(x)$ be?" δ is a response: "this close to a is sufficient." The definition says: no matter how tight the challenge, we can always find a sufficient response.

Example 5.4 — ε-δ proof for a linear function

Prove $\lim_{x\to 3}(2x + 1) = 7$ using the ε-δ definition.

Proof. Given $\varepsilon > 0$. We need $|(2x + 1) - 7| < \varepsilon$, i.e. $|2x - 6| = 2|x - 3| < \varepsilon$. Choose $\delta = \varepsilon/2$.
For $0 < |x - 3| < \delta$:

$$|(2x + 1) - 7| = 2|x - 3| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Example 5.5 — ε-δ proof for x^2

Prove $\lim_{x\to 2} x^2 = 4$.

Proof. $|x^2 - 4| = |x - 2||x + 2|$. We need to bound $|x + 2|$ near $x = 2$. Restrict: $|x - 2| < 1$, i.e. $1 < x < 3$, so $|x + 2| < 5$.
Given $\varepsilon > 0$, choose $\delta = \min(1, \varepsilon/5)$. For $0 < |x - 2| < \delta$:

$$|x^2 - 4| = |x - 2||x + 2| < \delta \cdot 5 \le \frac{\varepsilon}{5} \cdot 5 = \varepsilon. \quad \square$$

Example 5.6 — Proving $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = a$

(i) For $f(x) = c$ (constant): $|f(x) - c| = 0 < \varepsilon$ for any δ. ✓
(ii) For $f(x) = x$: $|f(x) - a| = |x - a| < \varepsilon$. Choose $\delta = \varepsilon$. ✓

5.4 The Fundamental Trigonometric Limit: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Theorem 5.1. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ *and* $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0.$

Geometric Proof of $\lim_{x\to 0}(\sin x)/x = 1$

For $0 < x < \pi/2$, consider the unit circle (radius 1).
Compare three areas:

- Triangle OAP (where $A = (1, 0)$, $P = (\cos x, \sin x)$): area $= \frac{1}{2} \cdot 1 \cdot \sin x$.
- Circular sector OAP: area $= \frac{1}{2} \cdot 1^2 \cdot x = \frac{x}{2}$.
- Triangle OAT (where $T = (1, \tan x)$): area $= \frac{1}{2} \cdot 1 \cdot \tan x$.

The containment gives:

$$\frac{\sin x}{2} \le \frac{x}{2} \le \frac{\tan x}{2}.$$

Dividing by $\sin x/2$ (positive for $0 < x < \pi/2$):

$$1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}.$$

Taking reciprocals (reversing inequalities):

$$\cos x \le \frac{\sin x}{x} \le 1.$$

As $x \to 0^+$: $\cos x \to 1$ and $1 \to 1$. By Squeeze Theorem: $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$.
For $x \to 0^-$: $\frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$, so the limit is also 1. □

Proof of $\lim_{x\to 0}(1 - \cos x)/x = 0$:

$$\frac{1-\cos x}{x} = \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}.$$

As $x \to 0$: $\frac{\sin x}{x} \to 1$ and $\frac{\sin x}{1+\cos x} \to \frac{0}{2} = 0$. Product $\to 0$. □

Example 5.7 — Evaluating trig limits

(a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$. (b) $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$.

Solution. (a) Write $\frac{\sin 3x}{x} = 3 \cdot \frac{\sin 3x}{3x}$. As $x \to 0$, $3x \to 0$, so $\frac{\sin 3x}{3x} \to 1$. Limit $= 3$.
(b) $\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x} \to 1 \cdot 1 = 1$.

Example 5.8 — More trig limits

(a) $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{x}$. (b) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{x}$.

Solution. (a) $\frac{1-\cos 2x}{x} = \frac{1-\cos 2x}{2x} \cdot 2 \to 0 \cdot 2 = 0$.
(b) $\frac{\sin^2 x}{x} = x \cdot \left(\frac{\sin x}{x}\right)^2 \to 0 \cdot 1^2 = 0$.

5.5 Derivatives of Trigonometric Functions

5.5.1 Derivatives of $\sin x$ and $\cos x$ from First Principles

Theorem 5.2. $\dfrac{d}{dx}(\sin x) = \cos x$ *and* $\dfrac{d}{dx}(\cos x) = -\sin x$.

Derivation of $\frac{d}{dx}\sin x$

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \qquad \text{(addition formula)} \\
&= \lim_{h\to 0}\sin x\cdot\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\frac{\sin h}{h} \\
&= \sin x\cdot 0+\cos x\cdot 1=\cos x. \quad \square
\end{aligned}$$

Derivation of $\frac{d}{dx}\cos x$

$$\begin{aligned}
\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} &= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \cos x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}-\sin x\cdot\lim_{h\to 0}\frac{\sin h}{h}=\cos x\cdot 0-\sin x\cdot 1=-\sin x. \quad \square
\end{aligned}$$

5.5.2 Derivatives of the Other Trigonometric Functions

Theorem 5.3.

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x\tan x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

Derivation of $\frac{d}{dx}\tan x$

$\tan x = \sin x/\cos x$. Apply the Quotient Rule:

$$\frac{d}{dx}\tan x = \frac{\cos x\cdot\cos x-\sin x\cdot(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x. \quad \square$$

Derivation of $\frac{d}{dx}\sec x$

$\sec x = 1/\cos x$. Using $\frac{d}{dx}[1/g] = -g'/g^2$:

$$\frac{d}{dx}\sec x = \frac{-(-\sin x)}{\cos^2 x}=\frac{\sin x}{\cos^2 x}=\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}=\sec x\tan x. \quad \square$$

Example 5.9 — Differentiating trig functions with chain rule

(a) $\dfrac{d}{dx}\sin(x^2)$. (b) $\dfrac{d}{dx}\tan^2(3x)$. (c) $\dfrac{d}{dx}\sec(x^3)$.

Solution. (a) $\cos(x^2) \cdot 2x = 2x\cos(x^2)$.
(b) Let $u = \tan(3x)$; we need $\dfrac{d}{dx}u^2 = 2u \cdot u'$. $u' = \sec^2(3x) \cdot 3 = 3\sec^2(3x)$. Answer: $6\tan(3x)\sec^2(3x)$.
(c) $\sec(x^3)\tan(x^3) \cdot 3x^2 = 3x^2\sec(x^3)\tan(x^3)$.

Example 5.10 — A simplification

Show that $\dfrac{d}{dx}[\sec^2 x + \tan^2 x] = 4\sec^2 x \tan x$.

Solution. $\sec^2 x + \tan^2 x = (1 + \tan^2 x) + \tan^2 x = 1 + 2\tan^2 x$.

$$\frac{d}{dx}[1 + 2\tan^2 x] = 2 \cdot 2\tan x \cdot \sec^2 x = 4\tan x \sec^2 x.$$

Alternatively, differentiate term by term and verify the same result.

5.6 Optimization Problems

Intuition

Many real-world problems ask for the best design: the maximum area, minimum cost, shortest time. These are **applied extrema problems**. The key is to set up a function to optimise and a constraint to work with.

General Strategy:

1. Read carefully; draw a diagram.
2. Identify the **objective function** (what to maximise or minimise).
3. Identify the **constraint** (an equation linking the variables).
4. **Eliminate** one variable using the constraint.
5. Find the **critical points** of the resulting one-variable function.
6. Use FDT or SDT (or evaluate at boundary) to determine the optimum.
7. Check that your answer makes physical sense.

Example 5.11 — Fencing problem (classic)

A farmer has 200 m of fencing to enclose a rectangular field along a river (one long side needs no fence). Find the dimensions that maximise the area.

Solution. Let x = width (two sides) and y = length (one side, along river). Constraint: $2x + y = 200 \to y = 200 - 2x$. Objective: $A = xy = x(200 - 2x) = 200x - 2x^2$. $A'(x) = 200 - 4x = 0 \Rightarrow x = 50$. $A''(x) = -4 < 0$: local (and global) maximum. $y = 200 - 100 = 100$.
Maximum area $= 50 \times 100 = 5000$ m^2.

Example 5.12 — Nearest point on a curve

Find the point on the parabola $y = \sqrt{x}$ closest to the point $(1, 0)$.

Solution. Minimise $D^2 = (x-1)^2 + y^2 = (x-1)^2 + x$ (substituting $y^2 = x$; minimising D^2 is equivalent).
Let $f(x) = (x-1)^2+x = x^2-2x+1+x = x^2-x+1$. $f'(x) = 2x-1 = 0 \Rightarrow x = 1/2$. $f''(x) = 2 > 0$: local minimum. ✓
Point: $(1/2, \sqrt{1/2}) = (1/2, 1/\sqrt{2})$. Minimum distance: $\sqrt{(1/2-1)^2 + 1/2} = \sqrt{1/4 + 1/2} = \sqrt{3/4} = \frac{\sqrt{3}}{2}$.

Example 5.13 — Cylinder with fixed volume, minimum surface area

A cylindrical can must hold exactly 1 litre ($= 1000$ cm^3). Find the radius and height that minimise the total surface area.

Solution. $V = \pi r^2 h = 1000 \Rightarrow h = \dfrac{1000}{\pi r^2}$.
$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \dfrac{1000}{\pi r^2} = 2\pi r^2 + \dfrac{2000}{r}$.
$S'(r) = 4\pi r - \dfrac{2000}{r^2} = 0 \Rightarrow 4\pi r^3 = 2000 \Rightarrow r^3 = \dfrac{500}{\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$.
$h = \dfrac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$.
Optimal: height = diameter. This is why many cans have equal height and diameter (for the given volume, this minimises material).

Example 5.14 — Norman window

A Norman window consists of a rectangle topped by a semicircle. The total perimeter is P. Find the dimensions that maximise the window's area.

Solution. Let r = radius of semicircle; width $= 2r$; height of rectangle $= h$.
Perimeter: $2h + 2r + \pi r = P \Rightarrow h = \dfrac{P - 2r - \pi r}{2} = \dfrac{P - r(2+\pi)}{2}$.
Area: $A = 2rh + \dfrac{\pi r^2}{2} = r[P - r(2+\pi)] + \dfrac{\pi r^2}{2} = Pr - r^2(2+\pi) + \dfrac{\pi r^2}{2}$.
$A'(r) = P - 2r(2+\pi) + \pi r = P - r(4 + 2\pi - \pi) = P - r(4+\pi) = 0 \Rightarrow r = \dfrac{P}{4+\pi}$.

Example 5.15 — Inscribed rectangle in a semicircle

Find the rectangle of maximum area inscribed in a semicircle of radius R.

Solution. Place the semicircle above the x-axis with centre at origin: $x^2+y^2 = R^2$, $y \geq 0$. Rectangle: width $2x$, height $y = \sqrt{R^2 - x^2}$.
$A = 2x\sqrt{R^2 - x^2}$. $A^2 = 4x^2(R^2 - x^2) = 4R^2x^2 - 4x^4$. $\frac{d(A^2)}{dx} = 8R^2x - 16x^3 = 0 \Rightarrow$ $x^2 = R^2/2 \Rightarrow x = R/\sqrt{2}$. $y = R/\sqrt{2}$. Maximum area $= 2 \cdot \dfrac{R}{\sqrt{2}} \cdot \dfrac{R}{\sqrt{2}} = R^2$.

5.7 Linearization

Definition

The **linearization** (or **linear approximation**) of f at $x = a$ is:

$$L(x) = f(a) + f'(a)(x - a).$$

$L(x)$ is the equation of the tangent line at $(a, f(a))$. For x near a: $f(x) \approx L(x)$.

Intuition

Zooming in on a smooth curve at a point, the curve looks like a straight line. The tangent line is the "best" linear approximation near the point of tangency.

Example 5.16 — Linearization of $\sqrt{1+x}$ near $x = 0$

Find $L(x)$ for $f(x) = \sqrt{1+x}$ at $a = 0$.

Solution. $f(0) = 1$. $f'(x) = \dfrac{1}{2\sqrt{1+x}}$, so $f'(0) = \dfrac{1}{2}$.

$L(x) = 1 + \dfrac{1}{2}x$.

Approximation: $\sqrt{1.02} \approx 1 + \dfrac{1}{2}(0.02) = 1.01$. Exact: $\sqrt{1.02} \approx 1.00995$. Error < 0.0001.

Example 5.17 — Linearization of $\sin x$ near $x = \pi/4$

Find $L(x)$ for $f(x) = \sin x$ at $a = \pi/4$.

Solution. $f(\pi/4) = \dfrac{\sqrt{2}}{2}$. $f'(\pi/4) = \cos(\pi/4) = \dfrac{\sqrt{2}}{2}$.

$L(x) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(x - \dfrac{\pi}{4}\right)$.

Approximate $\sin(46°)$: $46° = \dfrac{46\pi}{180}$ rad, $\dfrac{\pi}{4} = \dfrac{45\pi}{180}$, difference $= \dfrac{\pi}{180}$. $\sin(46°) \approx \dfrac{\sqrt{2}}{2}\left(1 + \dfrac{\pi}{180}\right) \approx 0.719$.

Example 5.18 — Estimating $\tan(44°)$

Use linearization to estimate $\tan(44°)$.

Solution. $a = 45° = \pi/4$. $f(x) = \tan x$. $f(\pi/4) = 1$. $f'(\pi/4) = \sec^2(\pi/4) = 2$.

$L(x) = 1 + 2\left(x - \dfrac{\pi}{4}\right)$.

$44° = \dfrac{\pi}{4} - \dfrac{\pi}{180}$. So $x - \pi/4 = -\pi/180$.

$\tan(44°) \approx 1 + 2 \cdot \left(-\dfrac{\pi}{180}\right) = 1 - \dfrac{\pi}{90} \approx 1 - 0.0349 \approx 0.965$.

Exact: $\tan(44°) \approx 0.966$. Excellent approximation.

Example 5.19 — General binomial approximation

For small x: $(1+x)^n \approx 1+nx$.
Proof: $f(x)=(1+x)^n$; $f(0)=1$; $f'(x)=n(1+x)^{n-1}$; $f'(0)=n$. $L(x)=1+nx$. ✓
Examples: $(1.001)^{100} \approx 1+100(0.001) = 1.1$. (Exact: ≈ 1.105.) $\sqrt[3]{1.03} = (1.03)^{1/3} \approx 1+\frac{1}{3}(0.03) = 1.01$.

5.8 Differentials and Error Estimation

Definition

The **differential** of $y=f(x)$ is:

$$dy = f'(x)\,dx.$$

Here $dx = \Delta x$ is the change in x, and dy approximates the change in y. The actual change $\Delta y = f(x+dx) - f(x) \approx dy$ for small dx.

Example 5.20 — Comparing dy and Δy

For $y = x^2$, compute dy and Δy at $x=3$ with $dx = 0.1$.
Solution. $dy = 2x\,dx = 2(3)(0.1) = 0.6$. $\Delta y = (3.1)^2 - 3^2 = 9.61 - 9 = 0.61$. Error: $|dy - \Delta y| = 0.01 = (dx)^2$ (second-order error).

Example 5.21 — Error in area

The radius of a circle is measured to be 5 cm with possible error ± 0.1 cm. Estimate the error in the calculated area.
Solution. $A = \pi r^2$; $dA = 2\pi r\,dr = 2\pi(5)(0.1) = \pi \approx 3.14$ cm^2. Relative error: $dA/A = 2\,dr/r = 2(0.1)/5 = 0.04 = 4\%$.

Example 5.22 — Error in volume of a sphere

If a sphere has radius $r = 10$ cm with error ± 0.05 cm, estimate the error in volume. $V = \frac{4}{3}\pi r^3$; $dV = 4\pi r^2\,dr = 4\pi(100)(0.05) = 20\pi \approx 62.8$ cm^3.

5.9 Chapter Summary

Concept	**Key Statement**
Radian	$180° = \pi$ radians; arc $= r\theta$
$\lim(\sin x)/x$	$= 1$; proved via unit-circle area squeeze
Trig derivatives	$(\sin x)' = \cos x$; $(\tan x)' = \sec^2 x$; etc. (derived from definition)
ε-δ definition	$\forall\varepsilon > 0, \exists\delta > 0$: $0 < \lvert x-a\rvert < \delta \Rightarrow \lvert f(x)-L\rvert < \varepsilon$
Optimization	Set up objective + constraint; eliminate; find critical points
Linearization	$L(x) = f(a) + f'(a)(x-a)$; best linear approximation near a
Differential	$dy = f'(x)dx$; approximates Δy for small dx

Key Takeaway

The derivative $f'(a)$ is simultaneously: the slope of the tangent, the instantaneous rate of change, and the coefficient in the best linear approximation $L(x) = f(a) + f'(a)(x-a)$. All of calculus flows from this single idea.

5.10 Exercises

Level 1 — Recall and Understand

1. Convert $\dfrac{7\pi}{6}$ to degrees and $210°$ to radians. Find sin and cos of each.
2. State the ε-δ definition of $\lim_{x\to a} f(x) = L$ in your own words. Then write it symbolically.
3. Using the ε-δ definition, prove $\lim_{x\to 5}(4x-3) = 17$.
4. Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{x}$ and $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}$.
5. Write out the complete table of trig derivatives (all six functions). Derive $\frac{d}{dx}\csc x$ from the quotient rule.

Level 2 — Apply

6. Differentiate: (a) $f(x) = \sin^3(2x)$; (b) $g(x) = \tan(\sqrt{x})$; (c) $h(x) = x\cos(x^2)$.
7. Differentiate $f(x) = \dfrac{\sin x}{1+\cos x}$ using the quotient rule. Simplify.
8. Find the equation of the tangent to $y = \sin x$ at $x = \pi/6$.
9. Find all values of $x \in [0, 2\pi]$ where $f(x) = \sin x + \cos x$ has a horizontal tangent.
10. A 10 m long ladder leans against a wall. The foot slides away from the wall at 0.5 m/s. How fast is the top sliding down when the foot is 6 m from the wall? *(Set up using the Pythagorean theorem; differentiate implicitly.)*
11. Find the linearization of $f(x) = \sqrt[3]{1+x}$ near $a = 0$. Use it to approximate $\sqrt[3]{1.06}$.
12. Estimate $\tan(46°)$ using linearization at $a = \pi/4$.
13. A square sheet of cardboard has side s cm. Small squares of side x cm are cut from each corner and the sides are folded up to make an open box. Find x that maximises the volume.
14. Find the point on the line $y = 2x + 1$ closest to the point $(3, 0)$.
15. Find the area of the largest rectangle with base on the x-axis and upper vertices on the parabola $y = 4 - x^2$.

Level 3 — Prove

16. Prove $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ using the geometric squeeze argument with unit circle areas.
17. Prove $\lim_{x\to 0} \dfrac{1-\cos x}{x} = 0$ using the result from Exercise 16.
18. Prove $\dfrac{d}{dx}\cos x = -\sin x$ from first principles using the addition formula for cos.
19. Prove the derivative formula for $\cot x = \cos x/\sin x$ using the quotient rule.
20. Using the ε-δ definition, prove $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ implies $\lim_{x\to a}[f(x)+g(x)] = L+M$.

Level 4 — Challenge

21. Among all rectangles with area 100 cm^2, find the one with the smallest perimeter. Prove your answer is indeed a minimum.
22. A wire of length L is cut into two pieces. One piece is bent into a square; the other into a circle. Find the cut that (a) minimises the total enclosed area, (b) maximises the total area.
23. Prove that among all triangles inscribed in a fixed circle, the equilateral triangle has the maximum area.
24. Let f be differentiable. Prove that the error in the linearization satisfies $|f(x) - L(x)| \leq M|x-a|^2/2$ if $|f''| \leq M$ near a. *(Apply the MVT twice.)*
25. Using $\lim_{x\to 0}\sin x/x = 1$, prove that $\lim_{x\to 0}(\cos x - 1)/x^2 = -1/2$. *Hint: Multiply and divide by* $(1+\cos x)$ *to get a form involving* $(\sin x/x)^2$.

Quick Reference: Differentiation Rules

The tables below collect all differentiation rules covered in this course.

Basic Rules

Rule	**Formula**
Constant	$\frac{d}{dx}[c] = 0$
Power	$\frac{d}{dx}[x^n] = nx^{n-1} \quad \text{(any real } n\text{)}$
Constant Multiple	$\frac{d}{dx}[cf(x)] = cf'(x)$
Sum / Difference	$\frac{d}{dx}[f \pm g] = f' \pm g'$
Product	$\frac{d}{dx}[fg] = f'g + fg'$
Quotient	$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}$
Chain	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Derivatives

$\frac{d}{dx}[\sin x] = \cos x$	$\frac{d}{dx}[\cos x] = -\sin x$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\frac{d}{dx}[\cot x] = -\csc^2 x$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\frac{d}{dx}[\csc x] = -\csc x \cot x$

Key Limit Results

Limit	**Value**
$\lim_{x\to 0} \frac{\sin x}{x}$	1
$\lim_{x\to 0} \frac{1-\cos x}{x}$	0
$\lim_{x\to \infty} \frac{\sin x}{x}$	0
$\lim_{x\to 0} (1+x)^{1/x}$	e

Hints and Answers

For each exercise the hint points you toward the key idea or method. The answer gives the final result. For proof exercises, "key step" gives the main idea to make the proof work.

Chapter 1 — Functions and Limits

1. *Hint:* Set $x - 2 \geq 0$ and $x + 1 \neq 0$ simultaneously. *Answer:* $[2, \infty)$.

2. *Hint:* The denominator must be nonzero and the expression under the square root nonnegative. *Answer:* $[-1, 1] \setminus \{0\}$ (i.e., $[-1, 0) \cup (0, 1]$).

3. *Hint:* Substitute $g(x)$ into f. *Answer:* $(f \circ g)(x) = (2x + 1)^2$; domain $= \mathbb{R}$.

4. *Hint:* Check whether $f(-x) = f(x)$ or $f(-x) = -f(x)$. *Answer:* $x^3 - x$ is odd; $x^2 + 1$ is even; $x + 1$ is neither.

5. *Hint:* Apply each transformation in order: horizontal shift, then vertical scale. *Answer:* Graph of $y = 2(x - 3)^2$: parabola shifted right 3, stretched vertically by factor 2.

6. *Hint:* The average rate of change is $[f(1 + h) - f(1)]/h$; simplify by expanding. *Answer:* $2 + h$.

7. *Hint:* Factor $x^2 - 9 = (x - 3)(x + 3)$ and cancel. *Answer:* 6.

8. *Hint:* Multiply numerator and denominator by $\sqrt{x + 4} + 2$. *Answer:* $\frac{1}{4}$.

9. *Hint:* Evaluate from the left ($x \to 0^-$) and right ($x \to 0^+$) separately. *Answer:* Left limit $= -1$; right limit $= 1$; limit does not exist.

10. *Hint:* Divide numerator and denominator by x^2 (the highest power). *Answer:* $\frac{3}{2}$.

11. *Hint:* Bound $|\sin(1/x)| \leq 1$ and apply Squeeze. *Answer:* Limit $= 0$.

12. *Hint:* Multiply by $\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$. *Answer:* $\frac{1}{2\sqrt{2}}$.

13. *Hint:* Factor $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$ and cancel. *Answer:* 12.

14. *Hint:* Write $x/\sqrt{x^2 + 1} = 1/\sqrt{1 + 1/x^2}$ for $x > 0$. *Answer:* 1.

15. *Hint:* Consider $\sqrt{x^2} = |x| = -x$ for $x < 0$. *Answer:* -1.

16. *Hint:* Identify the dominant term in the numerator and denominator. *Answer:* 0.

17. *Hint:* As $x \to 2^-$, the numerator $\to 3 > 0$ and $(x-2)^2 \to 0^+$. *Answer:* $+\infty$.

18. *Hint:* Factor the denominator and test the sign on each side. *Answer:* Left limit $= +\infty$; right limit $= -\infty$.

19. *Key step:* For $\lim_{x\to a}[f(x)+g(x)] = L+M$: given $\varepsilon > 0$, choose $\delta = \min(\delta_1, \delta_2)$ where δ_1, δ_2 correspond to $\varepsilon/2$ for f and g respectively; then use the triangle inequality.

20. *Key step:* Write $|fg - LM| = |f(g-M) + M(f-L)| \le |f||g-M| + |M||f-L|$; bound $|f|$ by $|L|+1$ near a.

21. *Key step:* $f(-x) = \sum a_k(-x)^k$. Even-degree terms satisfy $(-x)^k = x^k$, odd-degree terms satisfy $(-x)^k = -x^k$. A polynomial with only odd-degree terms is odd.

22. *Key step:* Use the Squeeze Theorem: $-|x| \le x\sin(1/x) \le |x|$ for $x \ne 0$, and both $\pm|x| \to 0$.

23. *Hint:* Vertical shift up 1; sketch $y = |x|$ first, then add 1. *Answer:* Graph is V-shaped with vertex at $(0,1)$.

24. *Hint:* Set up $|2x+1-5| = |2(x-2)| < \varepsilon$; choose $\delta = \varepsilon/2$. *Key step:* $|f(x) - L| = 2|x-2| < 2\delta = \varepsilon$.

25. *Hint:* Use the Sandwich Theorem: show that for large x, $\frac{x^2-1}{x^2+1} \le \frac{x^2\sin^2(x)+x}{x^2+1} \le \frac{x^2+x}{x^2+1}$ or similar bounding. Alternatively, split and use $|\sin x| \le 1$. *Answer:* Limit $= 1$.

Chapter 2 — Continuity and Differentiation

1. *Hint:* Check: (i) $f(2)$ exists; (ii) $\lim_{x\to2} f(x)$ exists; (iii) they are equal. *Answer:* f is continuous at $x = 2$ only if the piecewise values match.

2. *Hint:* Factor $x^2 + x - 6 = (x+3)(x-2)$ and $x^2 - 4 = (x-2)(x+2)$. *Answer:* Removable discontinuity at $x = 2$; define $f(2) = 5/4$ to remove it.

3. *Hint:* Show $f(1) < 0$ and $f(2) > 0$ (or vice versa), then apply IVT. *Answer:* $f(1) = -1 < 0$, $f(2) = 5 > 0$; root exists in $(1,2)$.

4. *Hint:* Use the definition: $f'(a) = \lim_{h\to0}[f(a+h) - f(a)]/h$. *Answer:* $f'(x) = 2x$.

5. *Hint:* Rationalize: multiply by $\frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}$. *Answer:* $f'(x) = \frac{1}{2\sqrt{x}}$.

6. *Hint:* Find $f'(-1)$ first using the power/product rules, then use $y - f(-1) = f'(-1)(x-(-1))$. *Answer:* Tangent: $y = f'(-1)(x+1) + f(-1)$ (evaluate for specific f).

7. *Hint:* Compute the left-hand derivative $\lim_{h\to0^-} |h|/h = -1$ and right-hand derivative $= 1$. *Key step:* The two one-sided derivatives are unequal, so $f'(0)$ does not exist.

8. *Hint:* Find k so that the values and derivatives match at the join point. *Answer:* Set the function values equal and the derivatives equal at $x = 1$; solve the resulting system.

9. *Hint:* Apply the product rule to each pair. *Answer:* $f'(x) = (x^2+1)(2)+(2x)(2x-3)$.

10. *Hint:* Use the quotient rule directly. *Answer:* $f'(x) = \frac{(2x)(x-1)-(x^2+1)(1)}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2}$.

11. *Hint:* Set $v(t) = s'(t) = 0$ and solve. *Answer:* $s'(t) = 3t^2 - 6 = 0 \Rightarrow t = \sqrt{2}$.

12. *Hint:* Identify inner and outer functions; apply chain rule. *Answer:* $f'(x) = 5(2x+1)^4 \cdot 2 = 10(2x+1)^4$.

13. *Hint:* Differentiate both sides with respect to x; treat y as a function of x. *Answer:* $\dfrac{dy}{dx} = \frac{x}{25-x^2} \cdot (-1)$... (work shown in text).

14. *Hint:* Differentiate $2y = x^2 + \sin y$ implicitly; collect dy/dx terms. *Answer:* $\dfrac{dy}{dx} = \frac{2x}{2-\cos y}$.

15. *Hint:* Differentiate $x^2 - xy + y^2 = 7$; substitute $(x, y) = (-1, 2)$. *Answer:* $\dfrac{dy}{dx}\Big|_{(-1,2)} = 0$; normal is $x = -1$.

16. *Hint:* Use $y = x^{p/q}$ implies $y^q = x^p$; differentiate implicitly. *Answer:* $\dfrac{dy}{dx} = \frac{p}{q}x^{p/q-1}$.

17. *Hint:* Find where $f'(x) = 0$. *Answer:* $f'(x) = 2x - 3 = 0 \Rightarrow x = 3/2$.

18. *Hint:* Write $f(x) = (x^2+1)(3x-5)^{-1}$ and use quotient rule; or expand. *Answer:* (Evaluate using quotient rule.)

19. *Key step:* Write $f(b) - f(a) = \lim_{h\to 0}[f(a+h) - f(a)]/h \cdot (b-a)$... Actually use f differentiable $\Rightarrow f(x) - f(a) = f'(a)(x-a) + o(x-a)$; take $x \to a$ to get $\lim_{x\to a} f(x) = f(a)$.

20. *Key step:* Prove product rule from first principles: add and subtract $f(x+h)g(x)$ in the numerator of $[f(x+h)g(x+h) - f(x)g(x)]/h$.

21. *Key step:* Chain rule proof: let $\Delta u = g(x+h) - g(x)$; when $\Delta u \neq 0$, write the difference quotient as $[f(u+\Delta u) - f(u)]/\Delta u \cdot \Delta u/h$; handle the case $\Delta u = 0$ separately.

22. *Hint:* $(fg)' = f'g + fg'$; $(fgh)' = [(fg)h]' = (fg)'h + (fg)h'$. *Answer:* $(fgh)' = f'gh + fg'h + fgh'$.

23. *Hint:* Find second derivative and set $= 0$. *Answer:* $f''(x) = 6x - 2$; equals 0 at $x = 1/3$.

24. *Key step:* For a polynomial of odd degree, the leading term dominates: $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$ (for positive leading coefficient). By IVT, some value 0 must be attained.

25. *Hint:* Differentiate $x^3 + y^3 = 1$ implicitly; differentiate again using quotient rule. *Answer:* $y'' = -\frac{2x}{y^5}$ (after simplification).

Chapter 3 — Extreme Values and the Mean Value Theorem

1. *Hint:* $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$. *Answer:* Critical points at $x = \pm 1$; $f(1) = -1$ (local min), $f(-1) = 3$ (local max).

2. *Hint:* The interval $[-2, 3)$ is not closed; check whether the supremum is attained. *Answer:* Absolute min $= f(0) = -3$; absolute max does not exist (function approaches 6 as $x \to 3^-$).

3. *Hint:* Evaluate f at critical points ($f' = 0$ or undefined) and endpoints. *Answer:* Max $= f(-1) = 8$; min $= f(2) = -19$ (for $f = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$).

4. *Hint:* Verify the three hypotheses of Rolle's Theorem; find c explicitly. *Answer:* $f'(x) = 2x - 2 = 0$ at $x = 1 \in (0, 2)$.

5. *Hint:* Compute $[f(2) - f(0)]/(2 - 0)$ and set equal to $f'(c)$. *Answer:* $c = 1$ (for $f(x) = x^2 - 2x$).

6. *Hint:* $[f(3) - f(1)]/(3 - 1) = 4$; solve $f'(c) = 4$. *Answer:* $c = 2$.

7. *Key step:* For any $a < b$ in I, MVT gives $f(b) - f(a) = f'(c)(b - a) = 0$, so $f(b) = f(a)$.

8. *Hint:* $f'(x) = 3x^2 + 12 > 0$ always. *Answer:* f is strictly increasing on $[-3, 3]$; no local extrema; abs min $= f(-3) = -58$, abs max $= f(3) = 68$.

9. *Hint:* $f(x) = x^4 + 3x + 1$; check $f(-2)$ and $f(-1)$ for IVT; check f' for monotonicity. *Answer:* $f(-2) = 11 > 0$, $f(-1) = -1 < 0$ (IVT gives existence); $f'(x) = 4x^3 + 3 < 0$ on $[-2, -1]$ (monotonicity gives uniqueness).

10. *Hint:* Build a sign chart for f'. *Answer:* $f'(x) = x^2 - 4x + 3 = (x - 1)(x - 3)$; increasing on $(-\infty, 1)$, decreasing on $(1, 3)$, increasing on $(3, \infty)$.

11. *Hint:* At $x = 1$: f' changes from $+$ to $-$ (local max); at $x = 3$: f' changes from $-$ to $+$ (local min). *Answer:* Local max at $x = 1$; local min at $x = 3$.

12. *Hint:* Apply the Closed Interval Method; find all critical points of f in $(-2, 6)$. *Answer:* (Evaluate f at $x = -1, 3$ and endpoints $x = -2, 6$.)

13. *Key step:* $\sin x < x$ for $x > 0$: define $g(x) = x - \sin x$; then $g(0) = 0$ and $g'(x) = 1 - \cos x \geq 0$, so g is non-decreasing, hence $g(x) \geq 0$.

14. *Key step:* For increasing: take $x_1 < x_2$ in the interval; MVT gives $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$; since $f'(c) > 0$ and $x_2 - x_1 > 0$, we get $f(x_2) > f(x_1)$.

15. *Hint:* $f'(x) = 4x^3 - 16x = 4x(x - 2)(x + 2)$. *Answer:* Sign chart shows local max at $x = 0$ (value 0), local min at $x = \pm 2$ (value -16).

16. *Key step:* Rolle's Theorem proof uses EVT to guarantee the maximum (or minimum) is attained at an interior critical point c; then the two-sided derivative argument gives $f'(c) = 0$.

17. *Key step:* MVT proof: define $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$; then $g(a) = g(b) = 0$; apply Rolle's Theorem to g.

18. *Hint:* $f'(x) = (2/3)x^{-1/3}$, which does not exist at $x = 0$; evaluate f at 0, -1, and 8. *Answer:* Max $= f(8) = 4$; min $= f(-1) = -1$.

19. *Hint:* $f'(x) = \frac{x^2+2x}{(x+1)^2}$; find critical points and test sign changes.

20. *Key step:* First Derivative Test proof: if f' changes from $-$ to $+$ at c, then f is decreasing just left of c and increasing just right of c, making $f(c)$ a local minimum.

21. *Hint:* By MVT, $|f(b) - f(a)| = |f'(c)||b-a| \leq M|b-a|$. *Answer:* $|f(4) - f(1)| \leq 2 \cdot 3 = 6$; so $f(4) \leq f(1) + 6 = 3 + 6 = 9$.

22. *Hint:* $\cos x < 1$ for $x \in (0, 2\pi)$ gives $\sin x - \sin 0 = \cos(c) \cdot x < x$. *Answer:* $|\sin x| \leq |x|$ for all x (use MVT with $a = 0, b = x$).

23. *Hint:* f continuous on $[-1, 8]$; $f'(x) = \frac{2}{3}x^{-1/3}$ does not exist at $x = 0$ (critical point); check $f(0), f(-1), f(8)$.

24. *Key step:* $f(x) = x^3 + x + 1$; $f'(x) = 3x^2 + 1 > 0$ everywhere; strictly monotone, so at most one real root. Since $f(0) = 1 > 0$ and $f(-1) = -1 < 0$, IVT gives exactly one root in $(-1, 0)$.

25. *Hint:* $f(a) = f(b)$ by assumption; apply Rolle's Theorem to f on $[a, b]$; the guaranteed c with $f'(c) = 0$ lies in (a, b).

Chapter 4 — Graphing and Asymptotes

1. *Hint:* Compute $f''(x)$; determine where it is positive and negative. *Answer:* Concave up on $(-\infty, 0)$; concave down on $(0, \infty)$; inflection point at $x = 0$.

2. *Hint:* $f''(x) = 12x^2 - 12 = 12(x-1)(x+1)$; check sign changes. *Answer:* Inflection points at $x = \pm 1$; concave up on $(-\infty, -1)$ and $(1, \infty)$.

3. *Hint:* $f'(x) = 4x^3 - 16x$; $f''(x) = 12x^2 - 16$; apply the Second Derivative Test at each critical point. *Answer:* Local min at $x = \pm 2$ ($f'' > 0$); local max at $x = 0$ ($f'' < 0$).

4. *Hint:* $f''(0) = 0$ but check whether f'' changes sign; $f^{(4)}(0) > 0$ suggests local min. *Answer:* SDT is inconclusive; use FDT: $f'(x) = 4x^3 > 0$ for $x > 0$ and < 0 for $x < 0$, so $x = 0$ is a local min.

5. *Hint:* Divide numerator and denominator by x (highest power); take $x \to \pm\infty$. *Answer:* $y = 2$ as $x \to \pm\infty$.

6. *Hint:* Divide by x^2 (the highest power). *Answer:* $y = 1/2$ as $x \to \pm\infty$.

7. *Hint:* Write $\frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{1+1/x^2}}$ for $x > 0$ and $\frac{-1}{\sqrt{1+1/x^2}}$ for $x < 0$. *Answer:* $y = 1$ as $x \to +\infty$; $y = -1$ as $x \to -\infty$.

8. *Hint:* $|\sin x| \le 1$, so $|\frac{\sin x}{x}| \le \frac{1}{|x|} \to 0$. *Answer:* $\lim_{x\to\infty} \frac{\sin x}{x} = 0$; the function $2 + \frac{\sin x}{x}$ has horizontal asymptote $y = 2$.

9. *Hint:* Factor $x^2 - 1 = (x-1)(x+1)$; cancel with numerator. *Answer:* $\lim_{x\to 1^+} = +\infty$ (check sign carefully); $\lim_{x\to -1^-} = -\infty$.

10. *Hint:* Perform polynomial long division: $\frac{x^2+1}{x} = x + \frac{1}{x}$. *Answer:* Oblique asymptote $y = x$ (as $\frac{1}{x} \to 0$); vertical asymptote $x = 0$.

11. *Hint:* Full analysis: domain, intercepts, symmetry, f', f'', asymptotes. *Answer:* $f(x) = (x^3+1)/x = x^2 + 1/x$; oblique asymptote $y = x^2$ (actually $y = x^2$ as a parabolic asymptote); vertical asymptote $x = 0$; local min at $x = 1$ ($f(1) = 2$); local max at $x = -1$ ($f(-1) = -2$).

12. *Hint:* $f'(x) = \frac{x(x-2)}{(x-1)^2}$; $f''(x) = \frac{2}{(x-1)^3}$. *Answer:* Vertical asymptote $x = 1$; oblique asymptote $y = x + 1$; local min at $x = 0$ ($f(0) = 0$); local max at $x = 2$ ($f(2) = 4$).

13. *Key step:* f'' changes sign at a point c means the curve's concavity switches there; this is an inflection point even if $f''(c) = 0$.

14. *Key step:* $f''(c) > 0$ means f' is increasing at c; since $f'(c) = 0$ and f' goes from negative to positive, $f(c)$ is a local minimum.

15. *Hint:* $f'(x) = 3x^2 - 6x = 3x(x-2)$; $f''(x) = 6x - 6$; SDT: $f''(0) = -6 < 0$ (local max at $x = 0$), $f''(2) = 6 > 0$ (local min at $x = 2$).

16. *Hint:* $f'(x) = (3x^2+3)(x-1) - (x^3+3x) \cdot 1$ (quotient); simplify.

17. *Hint:* The function $h(x) = f(x) - L$ has $h(\pm\infty) = 0$; so f approaches $L = 2$ from below (draw); the function oscillates, so the HA is approached from both sides. *Answer:* Yes; a function can cross its horizontal asymptote infinitely many times (e.g. $(\sin x)/x$).

18. *Hint:* For an oblique asymptote $y = mx + b$: $m = \lim_{x\to\infty} f(x)/x$ and $b = \lim_{x\to\infty}[f(x) - mx]$. *Answer:* (Apply the two-step formula to the given function.)

19. *Hint:* Use $f(0) = 1$ as a starting value; $f' > 0$ left of 0 means increasing; $f' < 0$ right of 0 means decreasing; $f \to 2$ at both ends means 2 is a HA. *Answer:* Graph has a peak at $(0, 1)$, rises from 2 on the left, dips, peaks, then returns to 2 on the right... Actually the description says $f \to 2$ and has max at 0: the graph looks like an inverted bell-curve above $y = 2$? Actually $f(0) = 1 < 2$: the graph rises from 2 (HA from left) through 1 (local max at 0) and back to 2 (HA at right). This describes a function with a local max below the asymptote.

20. *Key step:* Second Derivative Test proof: if $f'(c) = 0$ and $f''(c) > 0$, then $f''(c) = \lim_{h\to 0}[f'(c+h) - f'(c)]/h > 0$ means f' changes from negative to positive near c, so c is a local minimum by the First Derivative Test.

21. *Hint:* $f'(x) = 3x^2 + a$; for no real critical points, need $f' > 0$ always: require $a > 0$ or check discriminant of $3x^2 + a = 0$. *Answer:* $a > 0$ ensures $f' > 0$ everywhere; no critical points, so the function is strictly monotone with no local extrema.

22. *Key step:* $f(x) = x^{2/3}(6-x)^{1/3}$; use product rule and simplify; $f'(x)$ has the form $(4-x)/[x^{1/3}(6-x)^{2/3}]$ after simplification; critical points at $x = 4$ and $x = 0, 6$ (where f' is undefined).

23. *Hint:* Divide: $\frac{2x^2+3}{x+1} = 2x - 2 + \frac{5}{x+1}$. *Answer:* Oblique asymptote $y = 2x - 2$.

24. *Hint:* Check the sign of $f(x) -$ asymptote for large $|x|$ and near the VAs. *Answer:* $f(x) - (x+1) = \frac{-1}{x-1}$; this is negative for $x > 1$ and positive for $x < 1$, so the curve approaches from below for $x > 1$ and from above for $x < 1$.

25. *Key step:* A horizontal asymptote $y = L$ means $\lim_{x\to\pm\infty} f(x) = L$. It is possible for f to oscillate across $y = L$ and still approach it. This does NOT violate the definition. Example: $f(x) = (\sin x)/x$.

Chapter 5 — Trigonometry, Optimization, and Linearization

1. *Answer:* $\frac{5\pi}{4}$ radians; arc length $= 5\pi/4$ (for unit circle) or $5\pi r/4$ for radius r.

2. *Answer:* $\sin(75°) = \sin(45° + 30°) = \frac{\sqrt{6}+\sqrt{2}}{4}$.

3. *Key step:* Given $\varepsilon > 0$, we need $\delta > 0$ such that $|x-3| < \delta \Rightarrow |2x-6| < \varepsilon$. Since $|2x-6| = 2|x-3|$, choose $\delta = \varepsilon/2$.

4. *Key step:* $|x^2-4| = |x-2||x+2|$; bound $|x+2| \le 5$ for $|x-2| < 1$; then $\delta = \min(1, \varepsilon/5)$.

5. *Answer:* $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$. *Hint:* Multiply and divide by 3: $\frac{\sin 3x}{x} = 3 \cdot \frac{\sin 3x}{3x}$.

6. *Answer:* $\lim_{x\to 0} \frac{1-\cos 2x}{x} = 0$. *Hint:* Multiply by $\frac{1+\cos 2x}{1+\cos 2x}$; use $1 - \cos^2 = \sin^2$.

7. *Answer:* $f'(x) = 3\cos(3x)$.

8. *Answer:* $f'(x) = 2\tan(x)\sec^2(x)$.

9. *Answer:* $f'(x) = -\sin(\cos x) \cdot (-\sin x) = \sin(\cos x)\sin x$.

10. *Hint:* Use $\sec^2 x - \tan^2 x = 1$; differentiate to get 0. *Answer:* $f'(x) = 0$ (the expression is identically 1).

11. *Key step:* In the proof of $\frac{d}{dx}\sin x = \cos x$, the two limits needed are $\lim_{h\to 0}\frac{\sin h}{h} = 1$ (proved geometrically) and $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ (derived from the first using $\frac{\cos h-1}{h} = \frac{-\sin^2 h}{h(1+\cos h)}$).

12. *Key step:* $\tan x = \sin x/\cos x$; quotient rule gives $(\cos x \cdot \cos x - \sin x \cdot (-\sin x))/\cos^2 x = (\cos^2 x + \sin^2 x)/\cos^2 x = 1/\cos^2 x = \sec^2 x$.

13. *Hint:* Let $A =$ length along fence parallel to river; $B =$ total width. Constraint: $A + 2B = P$; maximize area $= AB$. *Answer:* Optimal: $A = P/2$, $B = P/4$; max area $= P^2/8$.

14. *Hint:* Minimize $D^2 = (x-1)^2 + y^2$ subject to $y = \sqrt{x}$, i.e. $y^2 = x$. *Answer:* Minimize $f(x) = (x-1)^2 + x$; $f'(x) = 2(x-1) + 1 = 0 \Rightarrow x = 1/2$; nearest point $= (1/2, 1/\sqrt{2})$.

15. *Hint:* Let $x =$ side of square base, $h =$ height; volume $= x^2h = V$ (fixed); minimize surface area $= 4xh + 2x^2$. *Answer:* Optimal: $h = x/2$; i.e., height equals half the base side.

16. *Hint:* Let $r =$ radius, $h =$ height of semicircle; window width $= 2r$; height of rectangle $= h$; perimeter $= 2h + 2r + \pi r = P$. *Answer:* Maximize $A = 2rh + \frac{\pi r^2}{2}$; eliminate h using the perimeter constraint.

17. *Hint:* $L(x) = f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x$. *Answer:* $\sqrt{1.02} \approx L(0.02) = 1.01$.

18. *Hint:* $f(x) = \sin x$, $a = \pi/4$; $L(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4)$. *Answer:* $\sin(46°) \approx L(46\pi/180)$.

19. *Hint:* $f(x) = \tan x$, $a = \pi/4$; $f'(x) = \sec^2 x$; $f'(\pi/4) = 2$. *Answer:* $L(x) = 1 + 2(x - \pi/4)$; for $x = 44\pi/180$: $\tan(44°) \approx 1 - 2 \cdot \frac{\pi}{180} \approx 0.9651$.

20. *Hint:* $dy = f'(x)\,dx$; for $f(x) = x^2$, $dy = 2x\,dx$. *Answer:* For $x = 3$, $dx = 0.1$: $dy = 2(3)(0.1) = 0.6$; exact change $\Delta y = 9.61 - 9 = 0.61$.

21. *Hint:* $A = \pi r^2$; $dA = 2\pi r\,dr$. *Answer:* $dA = 2\pi(5)(0.1) = \pi \approx 3.14$ sq. units.

22. *Hint:* $V = \frac{4}{3}\pi r^3$; $dV = 4\pi r^2\,dr$. *Answer:* $dV = 4\pi(10)^2(0.05) = 20\pi \approx 62.8$ cubic units.

23. *Key step:* $\frac{d}{dx}(x^{p/q})$: let $y = x^{p/q}$; then $y^q = x^p$; differentiating implicitly: $qy^{q-1}y' = px^{p-1}$; solve for $y' = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}x^{p/q-1}$.

24. *Key step:* $\lim_{h\to 0} \frac{\sin h}{h} = 1$ is proved by the squeeze: for $0 < h < \pi/2$, area of triangle $\leq$ area of sector $\leq$ area of larger triangle gives $\cos h \leq \frac{\sin h}{h} \leq \frac{1}{\cos h}$; as $h \to 0$, both bounds $\to 1$.

25. *Hint:* For $f(x) = x^3 - x + 1$: find f', set to zero to find critical points; evaluate f there and at endpoints of any bounded interval to find global extrema. *Answer:* $f'(x) = 3x^2 - 1 = 0$ at $x = \pm 1/\sqrt{3}$; local max $f(-1/\sqrt{3})$, local min $f(1/\sqrt{3})$; no global extrema on $\mathbb{R}$ (function is unbounded).

Appendix: October 2024 Examination Paper

This is the first external examination for MAT1CJ101 under the CUFYUGP 2024 regulations, sat by the first batch of B.Sc. Mathematics Honours students in October 2024.

Time: 2 hours **Maximum Marks:** 70

Section A Answer any questions; maximum 24 marks. [3 marks each]

1. Find the domain and range of $f(x) = \sqrt{9 - x^2}$.
2. Evaluate $\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$.
3. Find $\frac{dy}{dx}$ if $y = (x^2 + 1)^5$.
4. Using the chain rule, find $f'(x)$ where $f(x) = (3x^2 - 2x + 1)^4$.
5. Find $\frac{dy}{dx}$ if $2y = x^2 + \sin y$.
6. Find the equation of the normal to the curve $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$.
7. Find the absolute maximum and minimum values of $f(x) = x^2 - 3$ on $[-2, 3)$.
8. Prove that if $f'(x) = 0$ for all x in an interval I, then f is constant on I.
9. Give an example of a function on $[0, 1]$ that has no local maximum or minimum value at $x = 0$. Explain.
10. If $f(x) = \tan(2x)$, find $f'(x)$.
11. Sketch the graph of $y = 1 + \sqrt{x - 2}$, stating the domain.

Section B Answer any questions; maximum 36 marks. [6 marks each]

12. Find the domain and range of $f(g(x))$ where $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x+3}$.
13. Evaluate $\lim_{x \to 0} \frac{x}{\sqrt{1+x} - 1}$ and $\lim_{x \to \infty} \frac{3x^2 - 2x + 1}{2x^2 + x - 3}$.
14. Differentiate $f(x) = \frac{x^2 + 1}{x^3 - 2x}$.
15. Find the derivative of $f(x) = x^2 \sin(3x)$ using the product rule.
16. Show that $f(x) = x^4 + 3x + 1$ has exactly one zero in $[-2, -1]$.
17. Discuss the continuity of $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$ at $x = 2$. If the discontinuity is removable, define $f(2)$ to make f continuous.
18. Find the asymptotes of the curve $y = 2 + \frac{\sin x}{x}$.

19. Given that $\lim_{x\to 0} f(x) = 1$, $\lim_{x\to\infty} f(x) = 2$, $f(0) = 3$, $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$, sketch a possible graph of f.
20. (a) Find the intervals on which $f(x) = x^3 + 12x + 5$, $-3 \le x \le 3$, is increasing or decreasing, and find its extreme values.

    (b) Extend $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$ to be continuous at $x = 2$.

Section C Answer **any one** of the following. [10 marks]

21. Discuss the curve $y = \dfrac{x^3 + 1}{x}$ for symmetry, asymptotes, increase, decrease, extreme values, concavity, and inflection points. Sketch the curve.
22. (a) Find all relative extrema and points of inflection of $f(x) = x^4 - 8x^2 + 5$.

    (b) Prove that if f and g are differentiable on $[a, b]$ and $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ for some constant C.