Ambedkar College of Arts and Science, Wandoor

End Semester Examination, October 2024 (Sample Paper) — Answer Key

Differential Calculus — MAT1CJ101

B.Sc. Mathematics Honours (CUFYUGP 2024) — Semester I

Maximum Marks: 70 **Time: 2 Hours**

Section A: 3 marks each (ceiling 24). Section B: 6 marks each (ceiling 36). Section C: any one, 10 marks.

Section A (3 marks each; ceiling 24)

1. Composite $f \circ g$ with $f(x) = \sqrt{x}$, $g(x) = x + 1$. $(f \circ g)(x) = f(x+1) = \sqrt{x+1}$. Its domain consists of the x in $D(g) = \mathbb{R}$ with $g(x) \in D(f) = [0, \infty)$, i.e. $x + 1 \geq 0$:

$$D(f \circ g) = [-1, \infty).$$

Values: $g(3) = 4$, so $f(g(3)) = f(4) = \boxed{2}$; and $f(9) = 3$, so $f(f(9)) = f(3) = \boxed{\sqrt{3}}$.

2. Rationalise the numerator.

$$\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{(x+2) - 2}{x\left(\sqrt{x+2} + \sqrt{2}\right)} = \lim_{x \to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \boxed{\frac{\sqrt{2}}{4}}.$$

3. Horizontal tangents of $y = x^4 - 2x^2 + 2$. A horizontal tangent needs $y' = 0$: $y' = 4x^3 - 4x = 4x(x-1)(x+1) = 0 \Rightarrow x = 0, \pm 1$. **Yes**, there are three, at the points $\boxed{(0,2),\ (1,1),\ (-1,1)}$.

4. Find a, b, c.

- The curve passes through the origin (the point of tangency): $y(0) = c = 0$.
- It is tangent to $y = x$ at the origin, so the slope there is 1: $y' = 2ax + b$, giving $y'(0) = b = 1$.
- It passes through $(1, 2)$: $a + b + c = 2$, giving $a = 1$.

So $\boxed{a = 1,\ b = 1,\ c = 0}$, and the curve is $y = x^2 + x$.

5. $2y = x^2 + \sin y$. Differentiate implicitly:

$$2y' = 2x + (\cos y)\, y' \implies \boxed{\frac{dy}{dx} = \frac{2x}{2 - \cos y}} \qquad (2 - \cos y \geq 1 > 0).$$

6. Normal to $x^2 - xy + y^2 = 7$ at $(-1, 2)$. Check: $1 + 2 + 4 = 7$, so the point lies on the curve. Differentiate implicitly:

$$2x - (y + xy') + 2yy' = 0 \implies y' = \frac{y - 2x}{2y - x}.$$

At $(-1, 2)$: $y' = \dfrac{2 + 2}{4 + 1} = \dfrac{4}{5}$, so the tangent slope is $\frac{4}{5}$ and the normal slope is $-\frac{5}{4}$. The normal is

$$y - 2 = -\tfrac{5}{4}(x + 1) \quad \Longleftrightarrow \quad \boxed{5x + 4y = 3} \quad \left(y = -\tfrac{5}{4}x + \tfrac{3}{4}\right).$$

7. Absolute extrema of $f(x) = x^{2/3}$ on $[-2, 3)$. $f'(x) = \frac{2}{3}x^{-1/3}$ is never 0 and does not exist at $x = 0$, so the only critical point is $x = 0$. Values: $f(0) = 0$ and $f(-2) = \sqrt[3]{4} \approx 1.587$. As $x \to 3^-$, $f(x) \to \sqrt[3]{9} \approx 2.080$, but $x = 3$ is excluded.

- **Absolute minimum** 0 at $x = 0$, since $x^{2/3} \geq 0$.
- **No absolute maximum**: f is increasing on $[0, 3)$, so it gets arbitrarily close to $\sqrt[3]{9}$ but never reaches it, and every value it takes is less than $\sqrt[3]{9}$.

Note on the question. The interval is half-open, so the Max–Min Theorem does not apply. If the interval is read as $[-2, 3]$, the absolute maximum is $\sqrt[3]{9}$, attained at $x = 3$.

8. If $f'(x) = 0$ at each point of an interval I, then f is constant on I. *Proof.* Take any two points $x_1 < x_2$ in I. Since f' exists throughout I, f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$ (differentiable $\Rightarrow$ continuous). By the Mean Value Theorem there is a $c \in (x_1, x_2)$ with

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0 \quad \Longrightarrow \quad f(x_1) = f(x_2).$$

So f takes the same value at any two points of I, i.e. $f(x) = C$ on I. ■

9. Neither a local maximum nor a local minimum at 0. Take

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & 0 < x \leq 1, \\ 0, & x = 0. \end{cases}$$

In every interval $[0, \delta)$, f takes positive values (where $\sin \frac{1}{x} = 1$, i.e. at $x = \frac{2}{(4n+1)\pi}$) and negative values (where $\sin \frac{1}{x} = -1$, i.e. at $x = \frac{2}{(4n+3)\pi}$). So $f(0) = 0$ is neither $\geq$ nor $\leq$ all nearby values. Hence f has **neither a local maximum nor a local minimum at** 0. (A function that is monotone near an endpoint always has a local extremum there, so the example has to oscillate.)

10. $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$. By the definition, we must show that for every $\epsilon > 0$ there is a number M such that for all x,

$$x > M \implies \left| \frac{1}{x} - 0 \right| < \epsilon.$$

Given $\epsilon > 0$, choose $M = \dfrac{1}{\epsilon}$. If $x > M = \frac{1}{\epsilon} > 0$, then x is positive and

$$\left| \frac{1}{x} \right| = \frac{1}{x} < \frac{1}{M} = \epsilon.$$

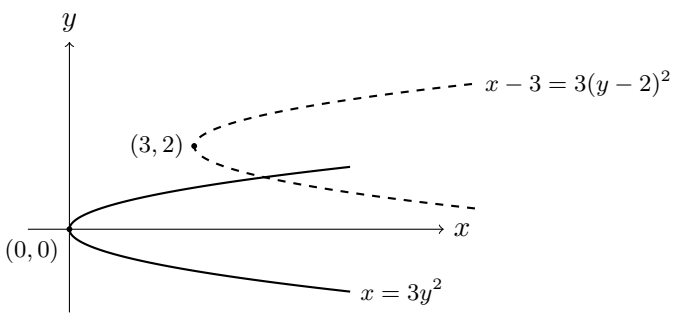
Hence $\lim_{x \to \infty} \frac{1}{x} = 0$. ■

Section B (6 marks each; ceiling 36)

11. Shift $x = 3y^2$ up 2 and right 3. Shifting up by 2 replaces y by $y - 2$, and shifting right by 3 replaces x by $x - 3$:

$$\boxed{x - 3 = 3(y - 2)^2} \qquad \text{i.e.} \qquad x = 3(y - 2)^2 + 3.$$

The original is a parabola opening to the right with vertex $(0, 0)$. The shifted copy has vertex $(3, 2)$ and axis $y = 2$. Points: $(3, 1)$, $(3, 3)$ on the original; $(6, 1)$, $(6, 3)$ on the shifted graph.



12. Is any real number exactly 1 less than its cube? We need $x = x^3 - 1$, i.e. a zero of $f(x) = x^3 - x - 1$. f is a polynomial, so it is continuous on $[1, 2]$. We have

$$f(1) = 1 - 1 - 1 = -1 < 0, \qquad f(2) = 8 - 2 - 1 = 5 > 0.$$

By the Intermediate Value Theorem (with value 0 between $f(1)$ and $f(2)$), there is a $c \in (1, 2)$ with $f(c) = 0$, i.e. $c = c^3 - 1$. **Yes**, such a number exists, and $c \approx 1.3247$. (It is in fact the only real one: $f'(x) = 3x^2 - 1$ shows the local maximum value of f, at $x = -1/\sqrt{3}$, is negative.)

13. Left-hand limit. *Definition.* Let $f(x)$ be defined on an interval (c, x_0), where $c < x_0$. If $f(x)$ gets arbitrarily close to a number M as x approaches x_0 from within that interval (that is, from the left, through values $x < x_0$), then f has **left-hand limit** M **at** x_0, and we write

$$\lim_{x \to x_0^-} f(x) = M.$$

Example. Let $f(x) = \dfrac{|x|}{x}$, which equals -1 for $x < 0$ and 1 for $x > 0$. Then

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1, \qquad \text{while} \qquad \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

The one-sided limits differ, so the two-sided limit at 0 does not exist. (A function has a limit at x_0 if and only if both one-sided limits exist and are equal.) Another example: $\lim_{x \to 1^-} \lfloor x \rfloor = 0$.

14. Average rate of change of $f(t) = 1/t$ from $t = 2$ to $t = 3$.

$$\frac{\Delta f}{\Delta t} = \frac{f(3) - f(2)}{3 - 2} = \frac{\frac{1}{3} - \frac{1}{2}}{1} = \boxed{-\frac{1}{6}}.$$

This is the slope of the secant line through $\left(2, \frac{1}{2}\right)$ and $\left(3, \frac{1}{3}\right)$. The negative sign shows f is decreasing on $[2, 3]$.

15. Implicit differentiation. *What it is.* Sometimes y is defined as a function of x only *implicitly*, by an equation $F(x, y) = 0$ that we cannot (or would rather not) solve for y. Implicit differentiation finds dy/dx in four steps:

(1) Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.

(2) Use the chain rule on every term containing y. For example, $\dfrac{d}{dx}(y^n) = ny^{n-1}\dfrac{dy}{dx}$.

(3) Collect the terms containing $\dfrac{dy}{dx}$ on one side.

(4) Solve for $\dfrac{dy}{dx}$.

When it is needed. Use it when y is hard or impossible to isolate (e.g. $y^5 + xy + x^3 = 3$), or when solving for y gives several branches (e.g. a circle, $y = \pm\sqrt{25 - x^2}$). It finds the slope at any point of the curve at once.

Example 1. $x^2 + y^2 = 25$: $2x + 2yy' = 0 \Rightarrow y' = -\dfrac{x}{y}$. At $(3, 4)$ the slope is $-\frac{3}{4}$.

Example 2. $y^2 = x^2 + xy$: $2yy' = 2x + y + xy' \Rightarrow y' = \dfrac{2x + y}{2y - x}$.

16. Exactly one zero of $f(x) = x^4 + 3x + 1$ in $[-2, -1]$. *At least one (IVT).* f is continuous, with $f(-2) = 16 - 6 + 1 = 11 > 0$ and $f(-1) = 1 - 3 + 1 = -1 < 0$. By the Intermediate Value Theorem, $f(c) = 0$ for some $c \in (-2, -1)$.

At most one. $f'(x) = 4x^3 + 3$. For $-2 \le x \le -1$ we have $x^3 \le -1$, so $f'(x) \le -4 + 3 = -1 < 0$. Thus f is strictly decreasing on $[-2, -1]$ and can cross zero only once. (Alternatively, by Rolle's Theorem: two zeros would force $f' = 0$ somewhere between them, which is impossible.)

Hence f has **exactly one** zero in $[-2, -1]$.

17. Asymptotes of $y = 2 + \dfrac{\sin x}{x}$ (Sandwich Theorem). For $x \neq 0$, $|\sin x| \leq 1$ gives

$$-\frac{1}{|x|} \leq \frac{\sin x}{x} \leq \frac{1}{|x|}.$$

Both bounds $\to 0$ as $x \to \pm\infty$, so by the Sandwich Theorem $\dfrac{\sin x}{x} \to 0$, and

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2.$$

So $\boxed{y = 2}$ is a horizontal asymptote on both the left and the right. The curve crosses it infinitely often, wherever $\sin x = 0$, $x \neq 0$.

There is no vertical asymptote. At $x = 0$, $\lim_{x\to 0} \frac{\sin x}{x} = 1$, so $y \to 3$ (a finite limit: a hole, not an asymptote).

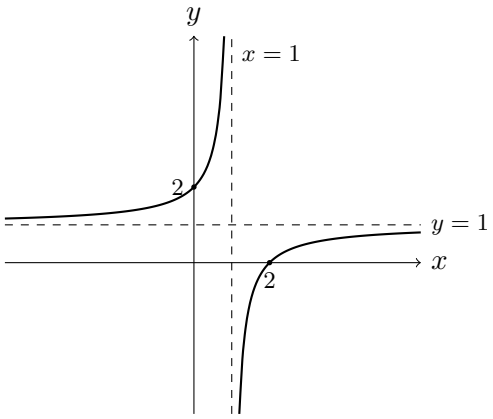
18. A function with the given limits. We want a horizontal asymptote $y = 1$ on both sides and a vertical asymptote $x = 1$, with $+\infty$ on the left and $-\infty$ on the right. Take

$$\boxed{f(x) = 1 + \frac{1}{1-x} = \frac{2-x}{1-x}}.$$

Check:

- $\frac{1}{1-x} \to 0$ as $x \to \pm\infty$, so $f \to 1$.
- As $x \to 1^-$, $1 - x \to 0^+$, so $f \to +\infty$.
- As $x \to 1^+$, $1 - x \to 0^-$, so $f \to -\infty$.

Other features: the y-intercept is $f(0) = 2$, the x-intercept is $x = 2$, and $f'(x) = \frac{1}{(1-x)^2} > 0$, so f is increasing on each branch.



Section C (answer any one; 10 marks)

19. (a) $f(x) = -x^3 + 12x + 5$, $-3 \leq x \leq 3$. $f'(x) = -3x^2 + 12 = -3(x-2)(x+2) = 0 \Rightarrow x = \pm 2$.

	$-3 < x < -2$	$-2 < x < 2$	$2 < x < 3$
f'	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$

f is **decreasing on** $[-3, -2]$ **and on** $[2, 3]$, and **increasing on** $[-2, 2]$.

Values: $f(-3) = 27 - 36 + 5 = -4$, $f(-2) = 8 - 24 + 5 = -11$, $f(2) = -8 + 24 + 5 = 21$, $f(3) = -27 + 36 + 5 = 14$.

- Local maximum -4 at the endpoint $x = -3$.
- Local minimum -11 at $x = -2$. This is the **absolute minimum**.
- Local maximum 21 at $x = 2$. This is the **absolute maximum**.
- Local minimum 14 at the endpoint $x = 3$.

(b) For $x \neq \pm 2$,

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x+3)(x-2)}{(x-2)(x+2)} = \frac{x+3}{x+2}, \qquad \text{so} \qquad \lim_{x \to 2} f(x) = \frac{5}{4}.$$

Since the limit at 2 exists, f has a continuous extension to $x = 2$:

$$F(x) = \begin{cases} \dfrac{x^2 + x - 6}{x^2 - 4}, & x \neq \pm 2, \\ \dfrac{5}{4}, & x = 2, \end{cases} \qquad \text{equivalently} \qquad F(x) = \frac{x+3}{x+2} \ (x \neq -2).$$

(At $x = -2$ the discontinuity is infinite and cannot be removed.)

20. Graph $y = \dfrac{x^3 + 1}{x} = x^2 + \dfrac{1}{x}$.

Domain and intercepts. The domain is $x \neq 0$. The x-intercept is $x^3 = -1 \Rightarrow x = -1$. There is no y-intercept.

Asymptotes and end behaviour.

- $\lim_{x \to 0^+} y = +\infty$ and $\lim_{x \to 0^-} y = -\infty$, so $x = 0$ is a vertical asymptote.
- As $x \to \pm\infty$, $y \to +\infty$, so there is no horizontal asymptote. The dominant term is x^2: for large $|x|$ the graph follows the parabola $y = x^2$.

First derivative. $y' = 2x - \dfrac{1}{x^2} = \dfrac{2x^3 - 1}{x^2}$, which is zero at $x = 2^{-1/3} \approx 0.794$.

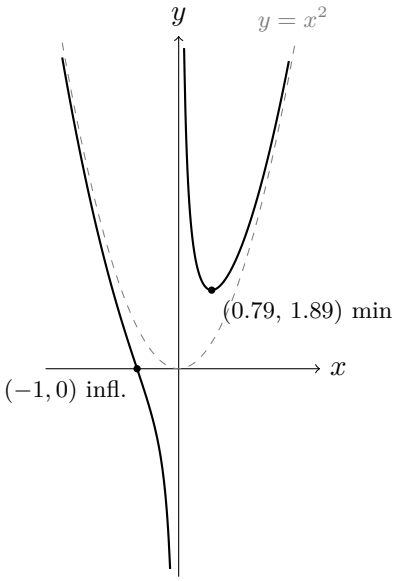
- $y' < 0$ on $(-\infty, 0)$ and on $(0, 2^{-1/3})$, so y is decreasing there.
- $y' > 0$ on $(2^{-1/3}, \infty)$, so y is increasing there.

So there is a **local minimum** at $x = 2^{-1/3}$, with value $y = 2^{-2/3} + 2^{1/3} = 3 \cdot 2^{-2/3} \approx 1.89$.

Second derivative. $y'' = 2 + \dfrac{2}{x^3} = \dfrac{2(x^3 + 1)}{x^3}$, which is zero at $x = -1$.

	$x < -1$	$-1 < x < 0$	$x > 0$
y''	$+$	$-$	$+$
concavity	up	down	up

So there is a **point of inflection at** $(-1, 0)$.



Sketch of $y = x^2 + \frac{1}{x}$: vertical asymptote $x = 0$, dominant term $y = x^2$ (grey, dashed).