

Ambedkar College of Arts and Science, Wandoor
First Semester (CUFYUGP) Degree Examination, November 2025 — Answer Key

Differential Calculus — MAT1CJ101

B.Sc. Mathematics Honours (CUFYUGP 2024) — Semester I

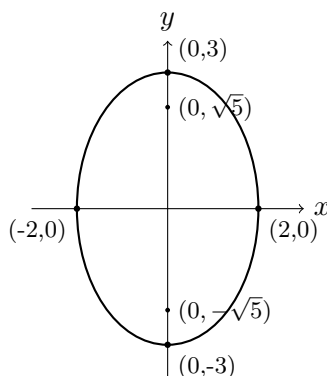
Maximum Marks: 70

Time: 2 Hours

Section A: 3 marks each (ceiling 24). Section B: 6 marks each (ceiling 36). Section C: any one, 10 marks.

Section A (3 marks each; ceiling 24)

1. **Graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$.** This is an ellipse centred at the origin. Setting $y = 0$ gives $x = \pm 2$, and setting $x = 0$ gives $y = \pm 3$. Since $9 > 4$, the **major axis is vertical** (along the y -axis), with vertices $(0, \pm 3)$, and the minor-axis ends are $(\pm 2, 0)$. The foci are at $(0, \pm c)$ with $c = \sqrt{9 - 4} = \sqrt{5}$. The curve is symmetric about both axes.



2. **Domain of $f(x) = \frac{1}{x(x-1)(x-2)}$.** The function is defined exactly when the denominator is non-zero, that is, when $x \neq 0, 1, 2$. Hence

$$D(f) = \mathbb{R} \setminus \{0, 1, 2\} = (-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, \infty).$$

3. **$\lim_{x \rightarrow 0^+} \ln \frac{2}{x^2}$.** As $x \rightarrow 0^+$, $x^2 \rightarrow 0^+$, so $u = \frac{2}{x^2} \rightarrow +\infty$. Since $\ln u \rightarrow +\infty$ as $u \rightarrow +\infty$,

$$\lim_{x \rightarrow 0^+} \ln \frac{2}{x^2} = \boxed{+\infty}$$

(the limit does not exist as a real number; the function grows without bound).

4. **Continuity of $\sqrt{2x+3}$.** The function is defined when $2x+3 \geq 0$, i.e. on $\left[-\frac{3}{2}, \infty\right)$. It is the composite of $g(x) = 2x+3$ (a polynomial, continuous everywhere) and $\sqrt{u}$ (continuous for $u \geq 0$). A composite of continuous functions is continuous, so f is continuous at every $x > -\frac{3}{2}$. At the left endpoint, $\lim_{x \rightarrow -3/2^+} \sqrt{2x+3} = 0 = f\left(-\frac{3}{2}\right)$, so f is right-continuous there. Hence f is **continuous on its whole domain $\left[-\frac{3}{2}, \infty\right)$.**

5. **Chain rule.** $\frac{dw}{dx} = \sec^2 x$ and $\frac{dx}{dt} = 12t^2 + 1$, so

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} = \boxed{(12t^2 + 1) \sec^2(4t^3 + t)}.$$

6. $y = \sin 3x$. By the chain rule, $y' = 3 \cos 3x$ and $\frac{d^2 y}{dx^2} = 3(-\sin 3x)(3) = \boxed{-9 \sin 3x}$ ($= -9y$).

7. **Absolute extrema of $f(x) = (x-2)^3$ on $[1, 4]$.** $f'(x) = 3(x-2)^2$, which is zero only at $x = 2$. Evaluate at the critical point and at the endpoints:

$$f(1) = -1, \quad f(2) = 0, \quad f(4) = 8.$$

Absolute maximum 8 at $x = 4$; **absolute minimum** -1 at $x = 1$. (Note that $f' \geq 0$, so f is increasing and the critical point $x = 2$ gives no extremum.)

8. **MVT for $f(x) = x^2 - x$ on $[-3, 5]$.** *Hypotheses:* f is a polynomial, so it is continuous on $[-3, 5]$ and differentiable on $(-3, 5)$. Both hypotheses hold. *Conclusion:* $f(5) = 20$ and $f(-3) = 12$, so

$$\frac{f(5) - f(-3)}{5 - (-3)} = \frac{20 - 12}{8} = 1.$$

Setting $f'(c) = 2c - 1 = 1$ gives $\boxed{c = 1}$, which does lie in $(-3, 5)$. It is the only such value.

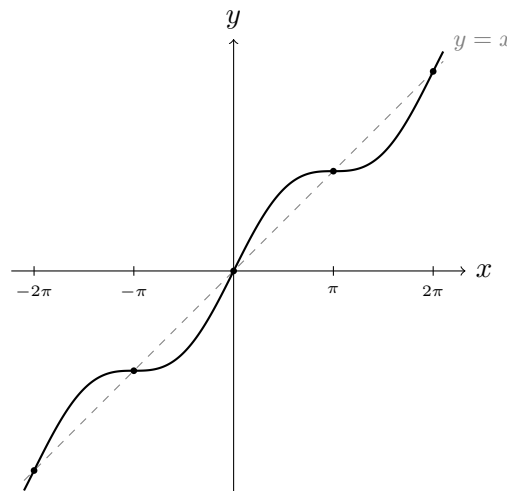
9. **Relative extrema of $f(x) = \frac{x+1}{x-1}$** (domain $x \neq 1$). By the quotient rule,

$$f'(x) = \frac{(x-1) - (x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0 \quad \text{for all } x \neq 1.$$

f' is never zero, and the only point where it fails to exist ($x = 1$) is not in the domain. So there are **no critical points** and hence **no relative extrema**. The function is decreasing on $(-\infty, 1)$ and on $(1, \infty)$, with vertical asymptote $x = 1$ and horizontal asymptote $y = 1$.

10. $f(x) = x + \sin x$. $f'(x) = 1 + \cos x \geq 0$, with equality only at the isolated points $x = (2k+1)\pi$. So f is **increasing on all of $\mathbb{R}$** and has **no relative extrema**. At $x = (2k+1)\pi$ the tangent is horizontal, but f' does not change sign.

$f''(x) = -\sin x$ changes sign at every $x = k\pi$. Concavity: concave down on $(2k\pi, (2k+1)\pi)$ and concave up on $((2k+1)\pi, (2k+2)\pi)$. So the **inflection points are $(k\pi, k\pi)$** , $k \in \mathbb{Z}$. They all lie on the line $y = x$, which the graph oscillates about.



Dots: inflection points $(k\pi, k\pi)$. At odd multiples of π the tangent is also horizontal.

Section B (6 marks each; ceiling 36)

11. **Given** $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 3$, **find** $\lim_{x \rightarrow 2} f(x)$. For $x \neq 2$ write

$$f(x) - 5 = \frac{f(x) - 5}{x - 2} \cdot (x - 2).$$

Both factors have limits as $x \rightarrow 2$, so the Product Rule gives

$$\lim_{x \rightarrow 2} (f(x) - 5) = \left(\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} \right) \left(\lim_{x \rightarrow 2} (x - 2) \right) = 3 \cdot 0 = 0.$$

Hence $\boxed{\lim_{x \rightarrow 2} f(x) = 5}$.

12. $\lim_{x \rightarrow 4} \frac{3 - x}{x^2 - 2x - 8}$. Factor the denominator: $x^2 - 2x - 8 = (x - 4)(x + 2)$. As $x \rightarrow 4$, the numerator tends to $3 - 4 = -1 \neq 0$ while the denominator tends to 0. So the quotient is unbounded and we look at the signs:

- $x \rightarrow 4^+$: $(x - 4) \rightarrow 0^+$ and $(x + 2) \rightarrow 6$, so the denominator $\rightarrow 0^+$ and the quotient $\rightarrow \frac{-1}{0^+} = -\infty$.
- $x \rightarrow 4^-$: the denominator $\rightarrow 0^-$, so the quotient $\rightarrow +\infty$.

The one-sided limits are $-\infty$ and $+\infty$, so the **limit does not exist**, and $x = 4$ is a vertical asymptote.

13. **Yes.** The quotient of continuous functions f/g is continuous at every point where $g \neq 0$ (Theorem on combinations of continuous functions). But if $g(c) = 0$ for some $c \in [0, 1]$, then f/g is not even defined at c , so it is discontinuous there.

Example: $f(x) = 1$ and $g(x) = x - \frac{1}{2}$ are both continuous on $[0, 1]$, but $\frac{f(x)}{g(x)} = \frac{1}{x - \frac{1}{2}}$ is discontinuous at $x = \frac{1}{2} \in [0, 1]$ (an infinite discontinuity).

Conclusion: f/g can be discontinuous on $[0, 1]$ exactly at the zeros of g . If g has no zero in $[0, 1]$, then f/g is continuous on $[0, 1]$.

14. **Implicit differentiation**, $\sqrt{y} = 3x^2 + \tan x$. Differentiate both sides with respect to x :

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = 6x + \sec^2 x \implies \boxed{\frac{dy}{dx} = 2\sqrt{y} (6x + \sec^2 x) = 2(3x^2 + \tan x)(6x + \sec^2 x)}.$$

(using $\sqrt{y} = 3x^2 + \tan x$). For the second derivative, differentiate $y' = 2\sqrt{y} (6x + \sec^2 x)$ with the product rule. Use $\frac{d}{dx} \sqrt{y} = \frac{y'}{2\sqrt{y}} = 6x + \sec^2 x$ and $\frac{d}{dx} \sec^2 x = 2 \sec^2 x \tan x$:

$$\frac{d^2 y}{dx^2} = 2(6x + \sec^2 x)^2 + 2\sqrt{y} (6 + 2 \sec^2 x \tan x),$$

that is,

$$\boxed{\frac{d^2 y}{dx^2} = 2(6x + \sec^2 x)^2 + 2(3x^2 + \tan x)(6 + 2 \sec^2 x \tan x)}.$$

(Check: $y = (3x^2 + \tan x)^2$, and differentiating this twice directly gives the same result.)

15. **Absolute extrema of $F(t) = 8t - t^4$ on $[-2, 1]$.** $F'(t) = 8 - 4t^3 = 0 \Rightarrow t^3 = 2 \Rightarrow t = \sqrt[3]{2} \approx 1.26$, which is *not* in $[-2, 1]$. So there are no interior critical points, and the extrema occur at the endpoints:

$$F(-2) = -16 - 16 = -32, \quad F(1) = 8 - 1 = 7.$$

Absolute maximum 7 at $t = 1$; **absolute minimum** -32 at $t = -2$. (Indeed $F' > 0$ on $[-2, 1]$, so F is increasing there.)

16. $f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}$.

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4(x - 1)}{3x^{2/3}}.$$

Critical points: $f'(x) = 0$ at $x = 1$, and f' does not exist at $x = 0$ (where f itself is defined). So the critical points are $\boxed{x = 0, x = 1}$.

Sign of f' : the denominator $3x^{2/3} > 0$ for $x \neq 0$, so f' has the sign of $x - 1$.

	$x < 0$	$0 < x < 1$	$x > 1$
f'	—	—	+
f	$\searrow$	$\searrow$	$\nearrow$

So f is **decreasing on** $(-\infty, 1]$ and **increasing on** $[1, \infty)$.

Extreme values: at $x = 0$, f' does not change sign, so there is no extremum (the curve has a vertical tangent there). At $x = 1$, f' changes from $-$ to $+$, so there is a **local minimum** $f(1) = 1 \cdot (-3) = -3$. This is also the **absolute minimum**, because f decreases up to $x = 1$ and increases afterwards. There is **no local or absolute maximum**, since $f(x) \rightarrow +\infty$ as $x \rightarrow \pm\infty$.

17. **Asymptotes of $f(x) = 2 + \frac{\sin x}{x}$.** *Horizontal.* For $x \neq 0$, $-\frac{1}{|x|} \leq \frac{\sin x}{x} \leq \frac{1}{|x|}$, and $\pm \frac{1}{|x|} \rightarrow 0$ as $x \rightarrow \pm\infty$. By the *Sandwich Theorem*, $\frac{\sin x}{x} \rightarrow 0$. Hence

$$\lim_{x \rightarrow \pm\infty} f(x) = 2 + 0 = 2,$$

so $\boxed{y = 2}$ is a horizontal asymptote, on both sides. The curve crosses this line infinitely often, at $x = k\pi$, $k \neq 0$.

Vertical. The only candidate is $x = 0$, where f is undefined. But $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, so $\lim_{x \rightarrow 0} f(x) = 3$ is finite. Hence $x = 0$ is **not** a vertical asymptote; it is a removable discontinuity.

Answer: the only asymptote is $y = 2$.

18. $\lim_{x \rightarrow -\infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$. Multiply the numerator and the denominator by x^4 :

$$\frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}} = \frac{x^3 + 1}{x^2 - x} = \frac{x + \frac{1}{x^2}}{1 - \frac{1}{x}}.$$

As $x \rightarrow -\infty$, the denominator $\rightarrow 1$ and the numerator $\rightarrow -\infty$. So the limit is $\boxed{-\infty}$. (The degree of the numerator exceeds that of the denominator by one, and the dominant term is x .)

Section C (answer any one; 10 marks)

19. $v(t) = t^2 - 4t + 3 = (t - 1)(t - 3)$ and $a(t) = \frac{dv}{dt} = 2t - 4$, for $t \geq 0$.

(a) $v = 0$ at $t = 1$ and $t = 3$. Then $a(1) = 2 - 4 = \boxed{-2}$ and $a(3) = 6 - 4 = \boxed{2}$ (in units of length/time²).

(b) The body moves forward when $v > 0$ and backward when $v < 0$.

t	$[0, 1)$	$(1, 3)$	$(3, \infty)$
$t - 1$	—	+	+
$t - 3$	—	—	+
v	+	—	+

It moves **forward for** $0 \leq t < 1$ **and for** $t > 3$, and **backward for** $1 < t < 3$. It is momentarily at rest at $t = 1$ and $t = 3$, where it changes direction.

(c) The velocity is increasing when $a = \frac{dv}{dt} > 0$, i.e. $2t - 4 > 0$, i.e. $t > 2$. It is decreasing when $a < 0$, i.e. $0 \leq t < 2$. The velocity reaches its minimum value $v(2) = -1$ at $t = 2$.

Interpretation: on $(1, 2)$ the body is moving backward and speeding up (speed $|v|$ increases). On $(2, 3)$ it is moving backward and slowing down.

20. (a) $f(x) = x^3 + \frac{4}{x^2} + 7$ on $(-\infty, 0)$.

Existence (IVT). f is continuous on $(-\infty, 0)$. We have

$$f(-3) = -27 + \frac{4}{9} + 7 < 0 \quad \text{and} \quad f(-1) = -1 + 4 + 7 = 10 > 0.$$

So by the Intermediate Value Theorem f has a zero in $(-3, -1)$. In fact $f(-2) = -8 + 1 + 7 = 0$, so the zero is $x = -2$.

Uniqueness (monotonicity).

$$f'(x) = 3x^2 - \frac{8}{x^3}.$$

For $x < 0$ we have $3x^2 > 0$ and $x^3 < 0$, so $-\frac{8}{x^3} > 0$. Hence $f'(x) > 0$ on $(-\infty, 0)$, and f is strictly increasing there. A strictly increasing function takes each value at most once. Therefore f has **exactly one zero** in $(-\infty, 0)$, namely $x = -2$.

- (b) $y' = (x-1)^2(x-2)(x-4)$.

Local extrema (First Derivative Test). The factor $(x-1)^2 \geq 0$ does not affect the sign of y' .

	$x < 1$	$1 < x < 2$	$2 < x < 4$	$x > 4$
y'	+	+	-	+

y' changes $+$ $\rightarrow$ $-$ at $x = 2$, giving a **local maximum at** $x = 2$. It changes $- \rightarrow +$ at $x = 4$, giving a **local minimum at** $x = 4$. At $x = 1$, $y' = 0$ but there is no sign change, so there is no extremum there.

Inflection points. Differentiate:

$$\begin{aligned} y'' &= 2(x-1)(x-2)(x-4) + (x-1)^2[(x-4) + (x-2)] \\ &= (x-1)[2(x^2 - 6x + 8) + (x-1)(2x-6)] \\ &= 2(x-1)(2x^2 - 10x + 11). \end{aligned}$$

$y'' = 0$ at $x = 1$ and at $x = \frac{10 \pm \sqrt{100 - 88}}{4} = \frac{5 \pm \sqrt{3}}{2}$, i.e. $x \approx 1.634$ and $x \approx 3.366$. These are three simple roots, so y'' changes sign at each:

	$x < 1$	$1 < x < \frac{5-\sqrt{3}}{2}$	$\frac{5-\sqrt{3}}{2} < x < \frac{5+\sqrt{3}}{2}$	$x > \frac{5+\sqrt{3}}{2}$
y''	-	+	-	+

So the graph of f has **points of inflection at** $x = 1$, $x = \frac{5-\sqrt{3}}{2}$ **and** $x = \frac{5+\sqrt{3}}{2}$. (At $x = 1$ the tangent is horizontal but there is no extremum: it is a horizontal point of inflection.)