Ambedkar College of Arts and Science, Wandoor

First Semester (CUFYUGP) Degree Examination, November 2024 — Answer Key

Differential Calculus — MAT1CJ101

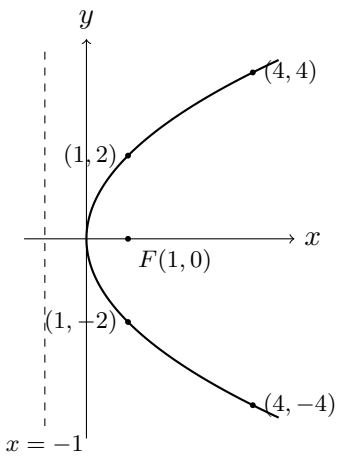
B.Sc. Mathematics Honours (CUFYUGP 2024) — Semester I

Maximum Marks: 70 **Time: 2 Hours**

Section A: 3 marks each (ceiling 24). Section B: 6 marks each (ceiling 36). Section C: any one, 10 marks.

Section A (3 marks each; ceiling 24)

1. Graph of $y^2 = 4x$. This is a parabola of the form $y^2 = 4ax$ with $a = 1$. It opens to the **right**, has vertex $(0, 0)$, axis the x-axis, focus $(1, 0)$ and directrix $x = -1$. It is symmetric about the x-axis (replacing y by $-y$ leaves the equation unchanged) and exists only for $x \geq 0$. Points: $(0, 0)$, $(1, \pm 2)$, $(4, \pm 4)$.



2. Three properties of $f(x) = |x|$, where $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$ (any three, with a line of justification each):

(i) $|x| \geq 0$ for every x, and $|x| = 0 \iff x = 0$. Hence the range is $[0, \infty)$.

(ii) $|-x| = |x|$: f is an *even* function and its graph is symmetric about the y-axis.

(iii) $|ab| = |a|\,|b|$ and $|a/b| = |a| \,/\, |b|$ $(b \neq 0)$.

(iv) Triangle inequality: $|a + b| \leq |a| + |b|$.

(v) $|x| < a \iff -a < x < a$ (for $a > 0$). Also $|x| = \sqrt{x^2}$.

(vi) f is continuous everywhere but not differentiable at $x = 0$ (corner: the left slope is -1 and the right slope is $+1$).

3. Limit at $+\infty$. Divide the numerator and the denominator by e^x:

$$\lim_{x \to \infty} \frac{1 - e^x}{1 + e^x} = \lim_{x \to \infty} \frac{e^{-x} - 1}{e^{-x} + 1} = \frac{0 - 1}{0 + 1} = \boxed{-1},$$

since $e^{-x} \to 0$ as $x \to \infty$.

4. Does $\lim_{x \to 1} \dfrac{1}{1 - x} = \lim_{x \to 0^-} \dfrac{1}{x}$ hold?

- As $x \to 1^-$, $1-x \to 0^+$, so $\dfrac{1}{1-x} \to +\infty$. As $x \to 1^+$, $1-x \to 0^-$, so $\dfrac{1}{1-x} \to -\infty$. The one-sided behaviours disagree, so $\lim_{x\to 1} \frac{1}{1-x}$ does **not exist**. It is not even an infinite limit.
- As $x \to 0^-$, $\dfrac{1}{x} \to -\infty$. This one-sided limit does not exist as a real number either. It is an infinite limit, written $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

Neither side is a real number, so the equation is **not meaningful**. The answer is **No**: the two sides do not even have the same infinite behaviour, because the left side does not tend to $-\infty$ from both directions.

5. **Continuity of $\dfrac{x}{\sin x}$.** The function is a quotient of two functions that are continuous everywhere. A quotient is continuous wherever its denominator is non-zero. Now $\sin x = 0 \iff x = k\pi$, $k \in \mathbb{Z}$. So $\dfrac{x}{\sin x}$ is **continuous at every** $x \neq k\pi$ ($k = 0, \pm 1, \pm 2, \dots$). It is undefined, and hence discontinuous, at $x = k\pi$.

   Remark. At $x = 0$ the discontinuity is removable, since $\lim_{x\to 0} \frac{x}{\sin x} = 1$. At $x = k\pi$ with $k \neq 0$ it is an infinite discontinuity.

6. **Composite and its derivative.** $(g \circ f)(x) = g(f(x)) = g(x^2) = \boxed{x^2 + 6}$.

   Chain rule: $(g \circ f)'(x) = g'(f(x))\, f'(x) = 1 \cdot 2x = \boxed{2x}$. Differentiating $x^2 + 6$ directly gives the same result.

7. **Absolute extrema of $f(x) = 4x^2 - 12x + 10$ on $[1, 2]$.** $f'(x) = 8x - 12 = 0 \Rightarrow x = \frac{3}{2} \in [1, 2]$. Evaluate f at the critical point and at the endpoints:

$$f(1) = 2, \qquad f\left(\tfrac{3}{2}\right) = 9 - 18 + 10 = 1, \qquad f(2) = 16 - 24 + 10 = 2.$$

   The **absolute maximum is** 2, at $x = 1$ and at $x = 2$. The **absolute minimum is** 1, at $x = \frac{3}{2}$.

8. **Intercepts and a zero of f'.** $x^2 - 5x + 4 = (x-1)(x-4) = 0 \Rightarrow x = 1,\ x = 4$. The function f is a polynomial, so it is continuous on $[1, 4]$ and differentiable on $(1, 4)$. Also $f(1) = f(4) = 0$, so *Rolle's Theorem* guarantees some $c \in (1, 4)$ with $f'(c) = 0$. Indeed $f'(x) = 2x - 5 = 0 \Rightarrow \boxed{c = \frac{5}{2}}$, which lies between 1 and 4.

9. **Critical points of $f(x) = x^3 - 3x^2$.** $f'(x) = 3x^2 - 6x = 3x(x-2)$. This derivative exists everywhere, so the critical points are exactly the zeros of f': $\boxed{x = 0,\ x = 2}$. A *stationary* point is a critical point at which $f' = 0$, as opposed to one where f' fails to exist. Here **both** critical points are stationary.

   (Extra: f' changes from $+$ to $-$ at 0 and from $-$ to $+$ at 2. So $f(0) = 0$ is a local maximum and $f(2) = -4$ is a local minimum.)

10. **"If the graph of f has a cusp at $x = 1$, then f cannot have an inflection point at $x = 1$." True.**

    At a cusp, $f'(x) \to +\infty$ on one side of 1 and $f'(x) \to -\infty$ on the other. Take the case $f'(x) \to -\infty$ as $x \to 1^-$ and $f'(x) \to +\infty$ as $x \to 1^+$.

    - On the left, f' becomes *unboundedly negative* as $x \to 1^-$. So f' cannot be increasing on any interval $(a, 1)$. The curve therefore cannot be concave up just to the left of 1.
    - On the right, f' is unboundedly large as $x \to 1^+$. So f' cannot be increasing on any interval $(1, b)$. The curve cannot be concave up just to the right either.

    So whatever concavity the curve has next to 1 is *concave down on both sides.* In the other case, $+\infty$ on the left and $-\infty$ on the right, the same argument gives concave up on both sides. The concavity does not change, so $x = 1$ is not an inflection point. (Example: $y = x^{2/3}$ has a cusp at 0 and is concave down on both sides.)

Section B (6 marks each; ceiling 36)

11. Centre and radius. Complete the squares:

$$(x^2+4x+4)+(y^2-6y+9)=3+4+9 \quad\Longrightarrow\quad (x+2)^2+(y-3)^2=16=4^2.$$

Comparing with $(x-h)^2+(y-k)^2=a^2$: the **centre is** $(-2,3)$ and the **radius is** 4.

12. "If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then so does $\lim_{x\to a}[f(x)+g(x)]$." True.

This is the *Sum Rule* for limits. If $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=M$ (both real numbers), then

$$\lim_{x\to a}[f(x)+g(x)]=L+M.$$

Intuitively, when $f(x)$ is close to L and $g(x)$ is close to M, the sum $f(x)+g(x)$ is close to $L+M$. For example, $\lim_{x\to 2} x^2=4$ and $\lim_{x\to 2} 3x=6$ give $\lim_{x\to 2}(x^2+3x)=10$.

Caution (the converse is false): the sum may have a limit even when neither summand does. Take $f(x)=\frac{1}{x}$ and $g(x)=-\frac{1}{x}$ at $a=0$. Then $f+g\equiv 0$ has limit 0, but neither $\lim_{x\to 0}\frac{1}{x}$ nor $\lim_{x\to 0}\left(-\frac{1}{x}\right)$ exists.

13. Continuous extension. For $t\neq 2$,

$$h(t)=\frac{t^2+3t-10}{t-2}=\frac{(t+5)(t-2)}{t-2}=t+5, \qquad \text{so} \qquad \lim_{t\to 2} h(t)=7.$$

Define $\boxed{h(2)=7}$. Then $h(2)$ is defined, $\lim_{t\to 2} h(t)$ exists, and the two are equal. So the extended function (which is just $t+5$) is continuous at $t=2$.

14. Implicit differentiation, $2y=x^2+\sin y$. Differentiate both sides with respect to x, treating y as a function of x:

$$2\frac{dy}{dx}=2x+\cos y\,\frac{dy}{dx} \;\Longrightarrow\; (2-\cos y)\frac{dy}{dx}=2x \;\Longrightarrow\; \boxed{\frac{dy}{dx}=\frac{2x}{2-\cos y}}.$$

The denominator is never zero, because $\cos y\le 1<2$.

15. Absolute extrema of $f(x)=\sqrt{x}+\cos x$ on $[0,2\pi]$. f is continuous on the closed interval $[0,2\pi]$, so absolute extrema exist (Extreme Value Theorem). They occur at critical points or at endpoints.

$f'(x)=\dfrac{1}{2\sqrt{x}}-\sin x$ for $x>0$. At $x=0$, f' does not exist, but $x=0$ is an endpoint anyway. Setting $f'(x)=0$ gives $\sin x=\dfrac{1}{2\sqrt{x}}$. This cannot be solved algebraically. Solving numerically (graphically or with Newton's method) gives two solutions in $(0,2\pi)$:

$$x_1\approx 0.662, \qquad x_2\approx 2.840.$$

For $x\in(\pi,2\pi)$ we have $\sin x<0<\frac{1}{2\sqrt{x}}$, so there are no further roots. Now compare the values:

x	0	0.662	2.840	2π
$f(x)$	1	≈ 1.602	≈ 0.730	$\sqrt{2\pi}+1\approx 3.507$

The **absolute maximum is** $\sqrt{2\pi}+1\approx 3.51$, at $x=2\pi$. The **absolute minimum is** ≈ 0.73, at $x\approx 2.84$.

16. Monotonicity of $f(x)=ax^2+bx+c$, $a\neq 0$. $f'(x)=2ax+b$, which is zero only at $x=-\dfrac{b}{2a}$. We use the First Derivative Test for monotonicity: f increases where $f'>0$ and decreases where $f'<0$.

- If $a > 0$: $f'(x) < 0$ for $x < -\frac{b}{2a}$ and $f'(x) > 0$ for $x > -\frac{b}{2a}$. So f is **decreasing on** $\left(-\infty, -\frac{b}{2a}\right]$ and **increasing on** $\left[-\frac{b}{2a}, \infty\right)$. The parabola opens upward, with a minimum at the vertex.
- If $a < 0$: the signs are reversed. f is **increasing on** $\left(-\infty, -\frac{b}{2a}\right]$ and **decreasing on** $\left[-\frac{b}{2a}, \infty\right)$. The parabola opens downward, with a maximum at the vertex.

Reasoning: f' is a linear function with slope $2a$, so it changes sign exactly once, at the vertex $x = -b/2a$. The sign of a decides which side is which.

17. Asymptotes of $f(x) = -\dfrac{8}{x^2 - 4}$.

Vertical. The denominator is $(x-2)(x+2) = 0$ at $x = \pm 2$, and the numerator $-8 \neq 0$ there. So $x = 2$ and $x = -2$ are vertical asymptotes:

$$\lim_{x\to 2^+} f = -\infty, \quad \lim_{x\to 2^-} f = +\infty, \quad \lim_{x\to -2^-} f = -\infty, \quad \lim_{x\to -2^+} f = +\infty.$$

For example, as $x \to 2^+$ we have $x^2 - 4 \to 0^+$, so $-8/(x^2-4) \to -\infty$.

Horizontal. $\lim\limits_{x\to\pm\infty} \dfrac{-8}{x^2-4} = \lim\limits_{x\to\pm\infty} \dfrac{-8/x^2}{1 - 4/x^2} = 0$, so $y = 0$ is a horizontal asymptote.

Answer: **vertical asymptotes $x = \pm 2$; horizontal asymptote $y = 0$.** (Also, f is even and $f(0) = 2$.)

18. Limit at infinity. Divide the numerator and the denominator by x, the highest power in the denominator:

$$\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 4} = \lim_{x\to\infty} \frac{\dfrac{2}{\sqrt{x}} + \dfrac{1}{x^2}}{3 - \dfrac{4}{x}} = \frac{0+0}{3-0} = \boxed{0}.$$

Section C (answer any one; 10 marks)

19. (a) We are given $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1} g(x) = 9$. The denominator's limit is $\lim_{x\to 1}(x - f(x)) = 1 - 2 = -1 \neq 0$, so the Quotient Rule applies, together with the Sum, Difference and Product Rules:

$$\lim_{x\to 1} \frac{f(x) - x^2 g(x)}{x - f(x)} = \frac{\lim f(x) - \left(\lim x^2\right)\left(\lim g(x)\right)}{\lim x - \lim f(x)} = \frac{2 - (1)(9)}{1 - 2} = \frac{-7}{-1} = \boxed{7}.$$

(b) $f(x) = \dfrac{3x+4}{x^2-4} = \dfrac{3x+4}{(x-2)(x+2)}$ is a rational function. A rational function is continuous at every point where its denominator is non-zero. So f is continuous on

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

At $x = \pm 2$, f is not defined, so it is discontinuous there. The numerator does not vanish at these points ($3(2) + 4 = 10$ and $3(-2) + 4 = -2$). So $|f(x)| \to \infty$ as $x \to \pm 2$, and both are **infinite discontinuities**. They cannot be removed, and $x = \pm 2$ are vertical asymptotes. For instance, $\lim_{x\to 2^+} f = +\infty$ and $\lim_{x\to 2^-} f = -\infty$.

20. (a) Yes. Since $f'(x) = 0$ for all x, f is differentiable, and hence continuous, everywhere. Take any $x \neq -1$ and apply the Mean Value Theorem to f on the interval between -1 and x. There is some c between them with

$$\frac{f(x) - f(-1)}{x - (-1)} = f'(c) = 0 \quad \Longrightarrow \quad f(x) = f(-1) = 3.$$

So $f(x) = 3$ for every x. (This is the corollary "$f' \equiv 0$ on an interval $\Rightarrow$ f is constant".)

(b) $y = x^2$: $y' = 2x$ and $y'' = 2 > 0$ for all x. Equivalently, $y' = 2x$ is increasing on $\mathbb{R}$. So the curve is **concave up** ("convex", holding water) on the whole real line $(-\infty, \infty)$. It is **nowhere concave down**, and since y'' never changes sign there are **no points of inflection**. The curve lies above each of its tangent lines and has its minimum at the vertex $(0, 0)$.