

Ambedkar College of Arts and Science, Wandoor

Class Test 2 (Diagnostic) — Solutions

Differential Calculus — MAT1CJ101 (Module II)

B.Sc. Mathematics Honours (CUFYUGP 2024) — Semester I

Maximum Marks: 30

Time: 30 Minutes

8 questions of 5 marks each; students answer any 6 (ceiling 30 marks).

Question 1. Continuity at a Point and on $\mathbb{R}$.

[5 Marks]

(a) For $x \neq 3$, $f(x) = \frac{(x-3)(x+2)}{x-3} = x+2$, so $\lim_{x \rightarrow 3} f(x) = 5$. Continuity at 3 requires (i) $f(3)$ defined, (ii) the limit exists, (iii) they are equal. Defining $f(3) = \boxed{5}$ satisfies all three.

(b) Each piece is a polynomial, hence continuous on its own interval; only the joints $x = 0$ and $x = 2$ need checking.

At $x = 0$: $\lim_{x \rightarrow 0^-} (2x + a) = a$ and $\lim_{x \rightarrow 0^+} (x^2 + 3) = 3 = g(0)$, so $a = 3$.

At $x = 2$: $\lim_{x \rightarrow 2^-} (x^2 + 3) = 7$ and $g(2) = \lim_{x \rightarrow 2^+} (bx - 1) = 2b - 1$, so $2b - 1 = 7$, $b = 4$.

$$\boxed{a = 3, b = 4.}$$

Question 2. Continuity on Intervals and the IVT.

[5 Marks]

(a) Let $f(x) = x^3 - 3x + 1$, a polynomial and so continuous everywhere. $f(0) = 1 > 0$, $f(1) = -1 < 0$, $f(2) = 3 > 0$. The *Intermediate Value Theorem* (if f is continuous on $[a, b]$ and N lies between $f(a)$ and $f(b)$, then $f(c) = N$ for some $c \in (a, b)$) with $N = 0$ gives a root in $(0, 1)$ since $f(0) > 0 > f(1)$, and another in $(1, 2)$ since $f(1) < 0 < f(2)$.

(b) For $-2 < x < 2$, f is a composition of continuous functions, so it is continuous at each such point. At the endpoints continuity means one-sided continuity: $\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0 = f(-2)$ and $\lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0 = f(2)$. Hence f is **continuous on** $[-2, 2]$.

(c) $h(x) = \frac{(x-1)(x+1)}{(x-1)(x-2)}$. At $x = 1$: for $x \neq 1$, $h(x) = \frac{x+1}{x-2} \rightarrow -2$, so the discontinuity is **removable** (define $h(1) = -2$). At $x = 2$: the numerator tends to $3 \neq 0$ while the denominator tends to 0, so $h(x) \rightarrow +\infty$ from the right and $-\infty$ from the left: an **infinite** discontinuity.

Question 3. The Derivative from the Definition.

[5 Marks]

(a)

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{2x+2h+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2x+2h+1} + \sqrt{2x+1}}{\sqrt{2x+2h+1} + \sqrt{2x+1}} = \frac{2h}{h(\sqrt{2x+2h+1} + \sqrt{2x+1})}.$$

Cancelling $h \neq 0$ and letting $h \rightarrow 0$:

$$\boxed{f'(x) = \frac{1}{\sqrt{2x+1}}.}$$

$f'(x)$ exists only when $2x+1 > 0$, i.e. domain $(-\frac{1}{2}, \infty)$; f is defined at $-\frac{1}{2}$ but has a vertical tangent there.

(b)

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \left(\frac{1}{x+h+2} - \frac{1}{x+2} \right) = \frac{1}{h} \cdot \frac{(x+2) - (x+h+2)}{(x+h+2)(x+2)} = \frac{-1}{(x+h+2)(x+2)}.$$

As $h \rightarrow 0$: $\boxed{g'(x) = -\frac{1}{(x+2)^2}}, \text{ for } x \neq -2.$

Question 4. Differentiability and Continuity.**[5 Marks]**

(a) $\lim_{x \rightarrow 1^-} x^2 = 1$ and $\lim_{x \rightarrow 1^+} (3x-2) = 1 = f(1)$, so f is continuous at 1. Left derivative: $\lim_{h \rightarrow 0^-} \frac{(1+h)^2 - 1}{h} = \lim_{h \rightarrow 0^-} (2+h) = 2$. Right derivative: $\lim_{h \rightarrow 0^+} \frac{3(1+h) - 2 - 1}{h} = 3$. They differ, so $f'(1)$ **does not exist**: the graph has a corner at $(1, 1)$. Continuity does not imply differentiability.

(b) Continuity at 1: $a + b = 1$. Differentiability also needs equal slopes: $2x|_{x=1} = 2 = a$. Hence $a = 2$, $b = -1$:

$$\boxed{a = 2, b = -1.}$$

(c) $\lim_{x \rightarrow 0} x^{1/3} = 0 = f(0)$, so y is continuous. But $\frac{f(h) - f(0)}{h} = \frac{h^{1/3}}{h} = h^{-2/3} \rightarrow +\infty$ as $h \rightarrow 0$ (from either side), so the graph has a vertical tangent at the origin and $f'(0)$ does not exist.

Question 5. Differentiation Rules and Tangent Lines.**[5 Marks]**

(a) Product rule, with $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$:

$$f'(x) = 2x(\sqrt{x} - 3x) + (x^2 + 1)\left(\frac{1}{2\sqrt{x}} - 3\right) = \boxed{\frac{5}{2}x^{3/2} - 9x^2 - 3 + \frac{1}{2\sqrt{x}}}.$$

(Check: $2x^{3/2} - 6x^2 + \frac{1}{2}x^{3/2} - 3x^2 + \frac{1}{2\sqrt{x}} - 3$.)

(b) Quotient rule:

$$g'(x) = \frac{2(x^2 + 1) - (2x - 3)(2x)}{(x^2 + 1)^2} = \frac{-2x^2 + 6x + 2}{(x^2 + 1)^2}.$$

At $x = 1$: $g'(1) = \frac{6}{4} = \frac{3}{2}$ and $g(1) = \frac{-1}{2}$. Tangent: $y + \frac{1}{2} = \frac{3}{2}(x - 1)$, i.e. $\boxed{y = \frac{3}{2}x - 2}$.

(c) $y' = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$ at $x = 1, 3$. $y(1) = 1 - 6 + 9 + 1 = 5$ and $y(3) = 27 - 54 + 27 + 1 = 1$. The points are $\boxed{(1, 5)}$ and $\boxed{(3, 1)}$.

Question 6. Rates of Change.**[5 Marks]**

(a) $v(t) = s'(t) = 3t^2 - 18t + 24$ m/s, $a(t) = v'(t) = 6t - 18$ m/s².

(b) $v(t) = 3(t^2 - 6t + 8) = 3(t-2)(t-4) = 0$: the particle is at rest at $\boxed{t = 2 \text{ s and } t = 4 \text{ s}}$.

(c) Distance is not displacement: the velocity changes sign at $t = 2$ and $t = 4$, so split the interval. $s(0) = 0$, $s(2) = 8 - 36 + 48 = 20$, $s(4) = 64 - 144 + 96 = 16$, $s(5) = 125 - 225 + 120 = 20$. Moves +20 on $[0, 2]$ (forward), -4 on $[2, 4]$ (back), +4 on $[4, 5]$ (forward). Total distance = $20 + 4 + 4 = \boxed{28 \text{ m}}$ (displacement is only 20 m).

(d) $v(1) = 3 - 18 + 24 = 9 > 0$ and $a(1) = -12 < 0$. Velocity and acceleration have opposite signs, so the speed $|v|$ is decreasing: the particle is **slowing down**.

Question 7. The Chain Rule.**[5 Marks]**

(a) With $u = \frac{2x+1}{x-3}$, $y = u^4$ gives $\frac{dy}{dx} = 4u^3 \frac{du}{dx}$. By the quotient rule,

$$\frac{du}{dx} = \frac{2(x-3) - (2x+1)}{(x-3)^2} = \frac{-7}{(x-3)^2}.$$

Hence

$$\boxed{\frac{dy}{dx} = 4 \left(\frac{2x+1}{x-3} \right)^3 \cdot \frac{-7}{(x-3)^2} = \frac{-28(2x+1)^3}{(x-3)^5}}.$$

(b) $(f \circ g)'(1) = f'(g(1)) g'(1) = f'(3) \cdot g'(1) = 4 \cdot 2 = \boxed{8}$.

$(g \circ f)'(1) = g'(f(1)) f'(1) = g'(5) \cdot f'(1) = 6 \cdot (-3) = \boxed{-18}$.

The outer derivative is always evaluated at the *inner* function's value.

Question 8. Implicit Differentiation and Rational Exponents.

[5 Marks]

(a) Differentiate with respect to x (the product xy needs the product rule): $2x + y + xy' + 2yy' = 0$, so $y' = -\frac{2x+y}{x+2y}$. At $(2, 1)$ (which lies on the curve, as $4 + 2 + 1 = 7$): $y' = -\frac{5}{4}$. Tangent:

$$y - 1 = -\frac{5}{4}(x - 2) \iff \boxed{5x + 4y = 14.}$$

(b) $y' = \frac{3}{2}(x^2 + 5)^{1/2} \cdot 2x = 3x\sqrt{x^2 + 5}$, so $y'(2) = 3 \cdot 2 \cdot 3 = \boxed{18}$.

(c) From $2x + 2yy' = 0$: $y' = -\frac{x}{y}$. Differentiate again (quotient rule, with $y' = -x/y$):

$$y'' = -\frac{y \cdot 1 - x y'}{y^2} = -\frac{y + x^2/y}{y^2} = -\frac{y^2 + x^2}{y^3} = \boxed{-\frac{25}{y^3}}$$

using $x^2 + y^2 = 25$.

— End of Solutions —