

Ambedkar College of Arts and Science, Wandoor

Class Test 1 — Solutions

Differential Calculus — MAT1CJ101 (Module I)

B.Sc. Mathematics Honours (CUFYUGP 2024) — Semester I

Maximum Marks: 30

Time: 30 Minutes

8 questions of 5 marks each; students answer any 6 (ceiling 30 marks).

Question 1. Composition and Domains.

[5 Marks]

(a) $(f \circ g)(x) = f(4 - x^2) = \sqrt{(4 - x^2) - 1} = \sqrt{3 - x^2}$. This needs $3 - x^2 \geq 0$, so the domain is $[-\sqrt{3}, \sqrt{3}]$. On this interval $3 - x^2$ takes every value in $[0, 3]$, so the range is $[0, \sqrt{3}]$.

(b) $(g \circ f)(x) = g(\sqrt{x - 1}) = 4 - (x - 1) = 5 - x$. The inner function requires $x \geq 1$, so the domain is $[1, \infty)$. Since $x \geq 1$, $5 - x \leq 4$ and every value ≤ 4 is attained: range $(-\infty, 4]$.

(c) The student is **wrong**. The simplified formula $5 - x$ looks defined everywhere, but the composite function inherits its domain from f : it is only defined for $x \geq 1$. The function $h(x) = 5 - x$ on $\mathbb{R}$ has a larger domain (and range $\mathbb{R}$), so $g \circ f \neq h$. Two functions are equal only when their domains and rules both agree.

Question 2. Even and Odd Functions.

[5 Marks]

(a) $F(x) = x^3 + 3x^2 + 3x + 1$. Grouping by parity of exponent,

$$F(x) = \underbrace{(3x^2 + 1)}_{\text{even}} + \underbrace{(x^3 + 3x)}_{\text{odd}}.$$

Check: $\frac{1}{2}[F(x) + F(-x)] = \frac{1}{2}[(x + 1)^3 + (1 - x)^3] = \frac{1}{2}[2 + 6x^2] = 3x^2 + 1$ and $\frac{1}{2}[F(x) - F(-x)] = x^3 + 3x$.

(b) $G(-x) = \lfloor -x \rfloor + \lfloor x \rfloor = G(x)$, so G is **even**. It is not odd: $G(\frac{1}{2}) = 0 + (-1) = -1$ and $G(-\frac{1}{2}) = -1 + 0 = -1$, so $G(-\frac{1}{2}) = G(\frac{1}{2}) \neq -G(\frac{1}{2}) = 1$. (For integer x , $G(x) = 0$; otherwise $G(x) = -1$.) Hence G is **even but not odd**.

Question 3. Transformations.

[5 Marks]

(a) f is defined for $-1 \leq u \leq 3$ with $u = 2x - 1$, so $-1 \leq 2x - 1 \leq 3$, i.e. $0 \leq x \leq 2$. On $[-1, 3]$, $f(u) = |u - 1|$ takes values in $[0, 2]$, so $g = 3 - 2f \in [3 - 4, 3 - 0] = [-1, 3]$.

(b) Writing $g(x) = 3 - 2f(2(x - \frac{1}{2}))$: compress horizontally by a factor 2, shift right by $\frac{1}{2}$, reflect in the x -axis, stretch vertically by a factor 2, shift up by 3.

(c) For $0 \leq x \leq 2$, $g(x) = 3 - 2|2x - 2| = 3 - 4|x - 1|$. Setting this to 0: $|x - 1| = \frac{3}{4}$, so $x = \frac{1}{4}$ or $x = \frac{7}{4}$, both in $[0, 2]$.

$$\boxed{x = \frac{1}{4}, \frac{7}{4}}.$$

Question 4. Evaluating Limits.

[5 Marks]

(a) Multiply by the conjugate:

$$\frac{\sqrt{x+5}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} = \frac{x-4}{(x-4)(\sqrt{x+5}+3)} = \frac{1}{\sqrt{x+5}+3} \xrightarrow{x \rightarrow 4} \frac{1}{6}.$$

(b) Combine the fractions in the numerator:

$$\frac{\frac{1}{(3+h)^2} - \frac{1}{9}}{h} = \frac{9 - (3+h)^2}{9h(3+h)^2} = \frac{-6h - h^2}{9h(3+h)^2} = \frac{-6 - h}{9(3+h)^2} \xrightarrow{h \rightarrow 0} \frac{-6}{81} = -\frac{2}{27}.$$

$$(c) \frac{x^3 + 1}{x^2 - 1} = \frac{(x+1)(x^2 - x + 1)}{(x-1)(x+1)} = \frac{x^2 - x + 1}{x-1} \text{ for } x \neq -1, \text{ which tends to } \frac{3}{-2} = -\frac{3}{2}.$$

Question 5. Greatest Integer Limits.**[5 Marks]**

(a) For x slightly less than 1, $x^2 \in (0, 1)$ so $\lfloor x^2 \rfloor = 0$; for x slightly greater than 1, $x^2 \in (1, 2)$ so $\lfloor x^2 \rfloor = 1$. The one-sided limits are 0 and 1, which differ, so the limit **does not exist**.

(b) For $0 < |x| < 1$, $0 \leq x^2 < 1$, so $\lfloor x^2 \rfloor = 0$ on both sides. The limit is $\boxed{0}$.

(c) By the defining property of the floor function, $\frac{1}{x} - 1 < \lfloor \frac{1}{x} \rfloor \leq \frac{1}{x}$.

- For $x > 0$, multiplying by x : $1 - x < x \lfloor \frac{1}{x} \rfloor \leq 1$.
- For $x < 0$, multiplying by x reverses the inequalities: $1 \leq x \lfloor \frac{1}{x} \rfloor < 1 - x$.

In both cases $x \lfloor \frac{1}{x} \rfloor$ is squeezed between 1 and $1 \pm |x|$, both tending to 1. By the squeeze theorem,

$$\lim_{x \rightarrow 0} x \lfloor \frac{1}{x} \rfloor = 1.$$

Question 6. Using the Limit Rules.**[5 Marks]**

(a) For $x \neq 1$,

$$f(x) - 5 = \frac{f(x) - 5}{x - 1} \cdot (x - 1).$$

As $x \rightarrow 1$ the first factor tends to 4 and the second to 0, so by the product rule $f(x) - 5 \rightarrow 4 \cdot 0 = 0$. By the sum rule, $f(x) = (f(x) - 5) + 5 \rightarrow 5$.

$$\lim_{x \rightarrow 1} f(x) = 5.$$

(The quotient is of the form 0/0 only after substitution; the quotient *rule* is not needed, and the hypothesis forces the numerator to vanish.)

(b) If the numerator did not tend to 0 as $x \rightarrow -1$ while the denominator does, the quotient would be unbounded and no finite limit could exist. So we need $(-1)^2 + k(-1) + 3 = 0$, i.e. $4 - k = 0$, $\boxed{k = 4}$. Then

$$\frac{x^2 + 4x + 3}{x + 1} = \frac{(x + 1)(x + 3)}{x + 1} = x + 3 \quad (x \neq -1) \rightarrow \boxed{2}.$$

(c) For $x \neq 0$, $\left| \frac{f(x)}{x} \right| \leq \frac{x^2}{|x|} = |x|$, so $-|x| \leq \frac{f(x)}{x} \leq |x|$. Both bounds tend to 0 as $x \rightarrow 0$, so by the Sandwich (Squeeze) Theorem

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0.$$

Question 7. Limits at Infinity.**[5 Marks]**

(a) For $x > 0$,

$$\sqrt{4x^2 + x} - 2x = \frac{(4x^2 + x) - 4x^2}{\sqrt{4x^2 + x} + 2x} = \frac{x}{\sqrt{4x^2 + x} + 2x} = \frac{1}{\sqrt{4 + 1/x} + 2} \xrightarrow{x \rightarrow \infty} \frac{1}{2 + 2} = \frac{1}{4}.$$

(b) As $x \rightarrow -\infty$, $\sqrt{4x^2 + x} \rightarrow +\infty$ and $-2x \rightarrow +\infty$; both terms are large and positive, so there is no cancellation:

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + x} - 2x) = +\infty.$$

(The conjugate trick of (a) is the trap here: for $x < 0$ the denominator $\sqrt{4x^2 + x} + 2x$ is a difference of large quantities of opposite size, and dividing by x without flipping the sign of $\sqrt{x^2} = |x| = -x$ wrongly gives $\frac{1}{4}$.)

(c) Divide numerator and denominator by x :

$$\frac{2x + \sin x}{x - \cos x} = \frac{2 + \frac{\sin x}{x}}{1 - \frac{\cos x}{x}}.$$

Since $|\sin x| \leq 1$, $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ for $x > 0$, and both bounds tend to 0; by the squeeze theorem $\frac{\sin x}{x} \rightarrow 0$, and likewise $\frac{\cos x}{x} \rightarrow 0$. Therefore the limit is $\frac{2+0}{1-0} = \boxed{2}$.

Question 8. True or False.

[5 Marks]

(a) **False.** $\frac{x^2 - 1}{x - 1}$ has limit 2 as $x \rightarrow 1$ but is undefined at 1. A limit describes behaviour *near* a , not at a .

(b) **True.** If $\lim(f + g)$ existed, then $g = (f + g) - f$ would have a limit by the difference rule, a contradiction.

(c) **False.** Let $f(x) = 1$ for $x > 0$ and 0 for $x \leq 0$, and $g(x) = 1 - f(x)$. Then $f(x)g(x) = 0$ for every x , so the limit of the product at 0 is 0, yet neither $\lim f$ nor $\lim g$ exists.

(d) **False.** $f(x) = 0$ and $g(x) = x^2$ at $a = 0$: $f(x) < g(x)$ for all $x \neq 0$, but both limits equal 0. Limits preserve $\leq$, not $<$.

(e) **False.** As $x \rightarrow 0^+$, $1/x \rightarrow +\infty$; as $x \rightarrow 0^-$, $1/x \rightarrow -\infty$. The two one-sided behaviours differ, so the two-sided limit does not exist (in particular it is not ∞).