

Ambedkar College of Arts and Science, Wandoor

Assignment 1 — Solutions

Differential Calculus — MAT1CJ101

B.Sc. Mathematics Honours (CUFYUGP 2024) — Semester I

Section A (3 marks each)

1. Domain of $f(x) = \frac{1}{\sqrt{x-2}(x+3)}$.

The square root needs $x - 2 \geq 0$, and it sits in the denominator, so $\sqrt{x-2} \neq 0$, i.e. $x > 2$. The factor $x + 3$ vanishes only at $x = -3$, which is already excluded.

$$\boxed{\text{Dom}(f) = (2, \infty)}$$

2. $f(x) = x^2$, $g(x) = x - 2$.

$$(f \circ g)(x) = f(x - 2) = (x - 2)^2 = x^2 - 4x + 4, \quad (g \circ f)(x) = g(x^2) = x^2 - 2.$$

They are equal only if $-4x + 4 = -2$, i.e. at $x = \frac{3}{2}$. As functions they are **not equal** (e.g. at $x = 0$: 4 versus -2).

3. $f(x) = \sqrt{x}$, $g(x) = x + 1$.

$$(f \circ g)(x) = \sqrt{x+1}, \text{ defined iff } x+1 \geq 0:$$

$$\text{Dom}(f \circ g) = [-1, \infty).$$

$$f(g(3)) = f(4) = \sqrt{4} = 2 \text{ and } f(9) = \sqrt{9} = 3.$$

4. Shift of $y = x^2$ by 4 right and 3 up.

Replace $x \rightarrow x - 4$ and add 3:

$$\boxed{y = (x - 4)^2 + 3}$$

(parabola with vertex $(4, 3)$).

5. $\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x - 3}$.

Factor: $x^2 - 2x - 3 = (x - 3)(x + 1)$. For $x \neq 3$ the fraction equals $x + 1$, so the limit is $3 + 1 = \boxed{4}$.

6. $\lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$.

Multiply by the conjugate:

$$\frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}} = \frac{(x+2) - 2}{x(\sqrt{x+2} + \sqrt{2})} = \frac{1}{\sqrt{x+2} + \sqrt{2}} \xrightarrow{x \rightarrow 0} \frac{1}{2\sqrt{2}}.$$

$$\boxed{\frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}}$$

7. Piecewise function at $x = 1$.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x + 1) = 3, \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2 + 2) = 3.$$

Both one-sided limits agree, so $\lim_{x \rightarrow 1} f(x)$ **exists and equals 3** (and $f(1) = 2(1) + 1 = 3$, so f is even continuous there).

8. $\lim_{x \rightarrow \infty} \frac{3x^3 - 2}{x^3 + 5x}.$

Divide numerator and denominator by x^3 :

$$\frac{3 - 2/x^3}{1 + 5/x^2} \rightarrow \frac{3 - 0}{1 + 0} = \boxed{3}.$$

9. Show $\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$

As x grows without bound, the denominator of $\frac{1}{x}$ becomes larger and larger while the numerator stays 1:

x	10	100	1000	10^4	10^6
$\frac{1}{x}$	0.1	0.01	0.001	0.0001	0.000001

So $\frac{1}{x}$ is always positive but gets smaller and smaller; whatever small positive number you name, $\frac{1}{x}$ is below it as soon as x is large enough (e.g. $\frac{1}{x} < 0.001$ for all $x > 1000$).

Hence $\frac{1}{x}$ gets arbitrarily close to 0, i.e. $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$. (This is the basic fact used in Q8: after dividing by the highest power of x , every term of the form $\frac{c}{x^n}$ tends to 0.) ■

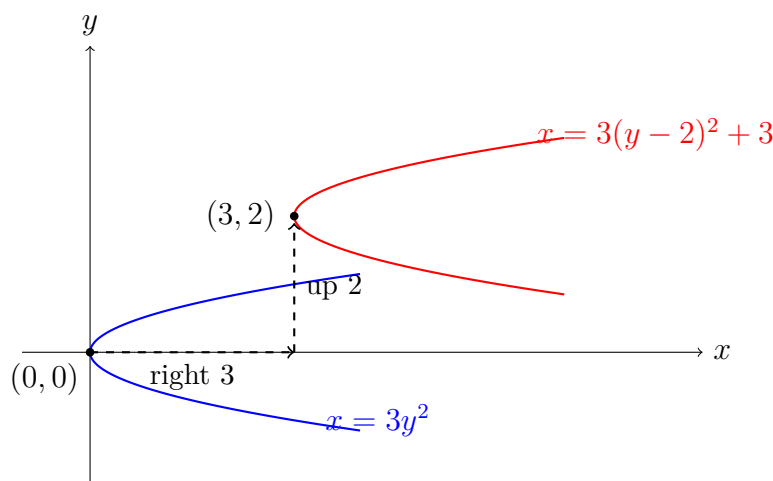
Section B (5 marks each)

1. Shift $x = 3y^2$ up 2 and right 3.

Moving up 2 replaces y by $y - 2$; moving right 3 replaces x by $x - 3$:

$$x - 3 = 3(y - 2)^2 \quad \Longleftrightarrow \quad \boxed{x = 3(y - 2)^2 + 3}.$$

The vertex moves from $(0, 0)$ to $(3, 2)$; the parabola still opens to the right. Check: the point $(3, 1)$ of the original ($3 = 3 \cdot 1^2$) moves to $(6, 3)$, and $3(3 - 2)^2 + 3 = 6$. ✓



2. Left-hand limit.

Definition. $\lim_{x \rightarrow x_0^-} f(x) = L$ means: $f(x)$ gets arbitrarily close to L as x approaches x_0 through values *less than* x_0 (that is, $x < x_0$ and $x \rightarrow x_0$). Only points to the *left* of x_0 are considered, and the value $f(x_0)$ itself plays no role.

Example. $f(x) = \frac{|x|}{x}$ ($x \neq 0$) at $c = 0$:

$$\lim_{x \rightarrow 0^-} f(x) = -1, \quad \lim_{x \rightarrow 0^+} f(x) = +1.$$

(Another standard example: $f(x) = \lfloor x \rfloor$ at any integer c .)

Geometric meaning. Approaching $x = c$ along the graph from the left, the curve heads to height -1 ; from the right it heads to height $+1$. The graph makes a **jump** at c , so there is no single height the curve approaches, and the two-sided limit $\lim_{x \rightarrow c} f(x)$ does not exist.

3. Asymptotes of $y = 2 + \frac{\sin x}{x}$.

For $x \neq 0$, $-1 \leq \sin x \leq 1$ gives $\left| \frac{\sin x}{x} \right| \leq \frac{1}{|x|}$, i.e.

$$-\frac{1}{|x|} \leq \frac{\sin x}{x} \leq \frac{1}{|x|}.$$

Since $\pm \frac{1}{|x|} \rightarrow 0$ as $x \rightarrow \pm\infty$, the Sandwich Theorem gives $\frac{\sin x}{x} \rightarrow 0$ and hence

$$\lim_{x \rightarrow \pm\infty} \left(2 + \frac{\sin x}{x} \right) = 2.$$

So $\boxed{y = 2}$ is a **horizontal asymptote** (on both sides).

Vertical asymptote? None: the only trouble point is $x = 0$, where $\frac{\sin x}{x} \rightarrow 1$, so $y \rightarrow 3$ (a removable “hole” at $(0, 3)$, not a blow-up). There is no oblique asymptote since the function tends to a finite constant. (Remark: the graph crosses $y = 2$ at $x = k\pi$, infinitely often — an asymptote only describes behaviour as $x \rightarrow \pm\infty$.)

Section C (8 marks)

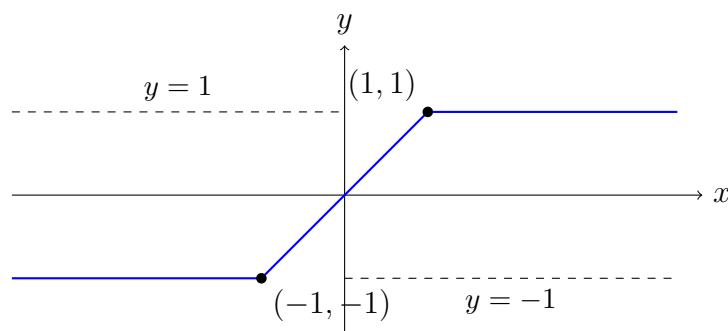
1. A function with $\lim_{x \rightarrow \infty} f = 1$, $\lim_{x \rightarrow -\infty} f = -1$, $\lim_{x \rightarrow 1^+} f = 1$.

(a) **Example.**

$$f(x) = \begin{cases} -1, & x \leq -1, \\ x, & -1 < x < 1, \\ 1, & x \geq 1. \end{cases}$$

Check: for $x \geq 1$, $f(x) = 1$, so $\lim_{x \rightarrow \infty} f = 1$ and $\lim_{x \rightarrow 1^+} f = 1$; for $x \leq -1$, $f(x) = -1$, so $\lim_{x \rightarrow -\infty} f = -1$. (An alternative, undefined at 0: $f(x) = x/|x|$.)

(b) **Sketch.** Horizontal ray at height -1 for $x \leq -1$, the line $y = x$ from $(-1, -1)$ to $(1, 1)$, then a horizontal ray at height 1 for $x \geq 1$.



- (c) **Can f be defined at $x = 1$?** Yes. The condition $\lim_{x \rightarrow 1^+} f(x) = 1$ only concerns values of f at points *near* 1 (to the right), never the value at 1 itself, so nothing prevents f from being defined there.
- (d) **Possible values of $f(1)$.** The limit is independent of $f(1)$, so $f(1)$ may be **any real number**; the stated conditions do not determine it. Examples: in the function above $f(1) = 1$ and f is continuous at 1. Changing the single value to $f(1) = 5$ keeps all three limit conditions true but creates a removable discontinuity (a jump of the point away from the curve). If one additionally demands continuity at 1, then $f(1)$ must equal $\lim_{x \rightarrow 1} f(x) = 1$ (which requires the left-hand limit to be 1 as well).