

Optimization Techniques for Machine Learning

AMLZC326 · #03 Linear Optimization I

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LEARNING OBJECTIVES

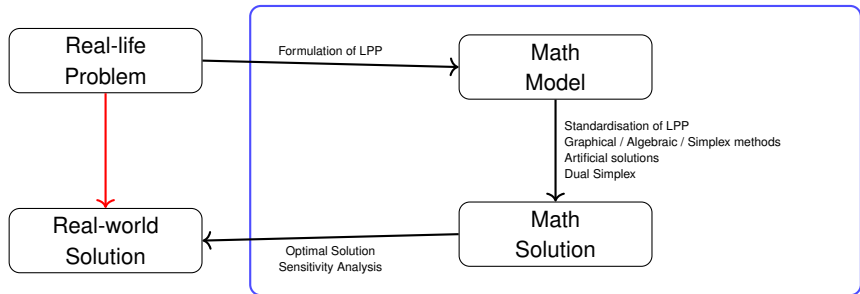
By the end of this lecture you should be able to:

- Formulate a real-world decision problem as a Linear Program (LPP) with an objective function and constraints
- Convert any LPP to standard form using slack and surplus variables
- Understand the corner-point theorem: why the optimum always occurs at a vertex of the feasible polytope
- Execute the Simplex Method step-by-step using the minimum ratio test and Gauss-Jordan pivot operations

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LINEAR PROGRAMMING PROBLEM (LPP)



FORMULATION OF AN LPP

- **Decision Variables:** Let x_1, x_2, \dots, x_n denote the unknown quantities to be determined. These represent the choices we can make (e.g., number of units of each product to manufacture).
- **Objective Function:** A linear function to be maximized or minimized:

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n.$$

(e.g., maximize profit $z = 8x_1 + 5x_2$).

- **Constraints:** A set of linear restrictions that limit our choices:

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i, \quad i = 1, 2, \dots, m.$$

Along with non-negativity conditions: $x_1, x_2, \dots, x_n \geq 0$.

REDDY MIKKS PAINT: PROBLEM STATEMENT

Reddy Mikks produces **exterior** and **interior** paints using two raw materials, **M1** and **M2**. The relevant data are given below:

	Exterior	Interior	Availability (tons/day)
Raw material M1	6	4	24
Raw material M2	1	2	6
Profit per ton (\$1000)	5	4	–

Additional demand conditions:

- Interior paint production cannot exceed exterior paint production by more than 1 ton.
- Maximum daily demand for interior paint is 2 tons.

Question: Determine the daily production levels of exterior and interior paints that maximize total profit.

REDDY MIKKS LPP FORMULATION

Decision Variables

x_1 = tons of exterior paint produced per day

x_2 = tons of interior paint produced per day

Objective Function

Maximize total daily profit (in thousands of dollars):

$$\text{Maximize } z = 5x_1 + 4x_2$$

Constraints

$$6x_1 + 4x_2 \leq 24 \quad (\text{Raw material M1})$$

$$x_1 + 2x_2 \leq 6 \quad (\text{Raw material M2})$$

$$-x_1 + x_2 \leq 1 \quad (\text{Demand relation})$$

$$x_2 \leq 2 \quad (\text{Interior demand limit})$$

$$x_1, x_2 \geq 0 \quad (\text{Non-negativity})$$

GEOMETRIC VIEW OF AN LPP

- Each linear constraint defines a **half-space** in \mathbb{R}^n .
- The **feasible region** is the intersection of all these half-spaces.
- A **feasible solution** is any point that satisfies *all* constraints.
- An **infeasible solution** violates at least one constraint.
- The objective is to find the **optimal solution**: the feasible solution that gives the best value of the objective function.
- Fundamental idea: **If an optimal solution exists, it occurs at a corner point (extreme point) of the feasible region.**

Graphical Method: For problems with two variables, plot all constraints to obtain the feasible region, and identify the optimal solution at one of its corner points.

STANDARD FORM OF AN LPP

An LPP is said to be in **standard form** if it satisfies the following conditions:

Objective Function:

The objective function must be in **maximization form**.

Constraints:

- All constraints are written as **equations**.
- The right-hand side (RHS) of each constraint is **nonnegative**.
- All variables are **nonnegative**.

Why standard form? The simplex method requires this structure to move systematically between feasible solutions.

CONVERTING AN LPP TO STANDARD FORM

- If the original problem is Minimize w convert it to a maximization problem by defining $z = -w$ and maximizing z .

Note: If z^ is the optimal value of the maximization problem, then the optimal value of the original minimization problem is $w^* = -z^*$.*

- (\leq) constraints: add a slack variable

$$6x_1 + 4x_2 \leq 24 \Rightarrow 6x_1 + 4x_2 + s_1 = 24, \quad s_1 \geq 0$$

- (\geq) constraints: subtract a surplus variable

$$x_1 + x_2 \geq 800 \Rightarrow x_1 + x_2 - s_2 = 800, \quad s_2 \geq 0$$

- If the RHS is negative, multiply the entire equation by -1 .
- If a variable is unrestricted in sign, replace it by the difference of two nonnegative variables: $x = x^+ - x^-$, $x^+ \geq 0$, $x^- \geq 0$

After applying these steps, the LPP is in standard form.

STANDARD FORM: REDDY–MIKKS PROBLEM

Objective Function: Maximize $z = 5x_1 + 4x_2$

Constraints:

$$6x_1 + 4x_2 + s_1 = 24$$

$$x_1 + 2x_2 + s_2 = 6$$

$$-x_1 + x_2 + s_3 = 1$$

$$x_2 + s_4 = 2$$

Non-negativity: $x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

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GRAPHICAL VS ALGEBRAIC SOLUTION METHODS

Graphical Method

Graph all constraints, including nonnegativity restrictions.
Solution space consists of infinitely many **feasible points**.

Identify **feasible corner points** of the solution space.
Candidates for the optimum are given by a **finite number of corner points**.

Use the objective function to determine the **optimum corner point** from among all candidates.

Algebraic Method

Represent the solution space by m equations in n variables and restrict all variables to non-negative values, $m < n$.
The system has infinitely many **feasible solutions**.

Determine the **feasible basic solutions** of the equations.
Candidates for the optimum are given by a **finite number of basic feasible solutions**.

Use the objective function to determine the **optimum basic feasible solution** from among all candidates.

BASIC AND BASIC FEASIBLE SOLUTIONS

In a system of m equations and n variables ($m < n$):

- Choose m variables as **basic variables** and set the remaining $n - m$ variables (nonbasic) to zero.
- Solve the m equations for the basic variables.
- The resulting solution is a **basic solution**.
- If all variables are ≥ 0 , it is a **basic feasible solution**.

Maximum number of basic solutions: $\binom{n}{m}$

EXAMPLE: CORNER POINTS AND BASIC FEASIBLE SOLUTIONS

Consider the following LPP with two variables:

Maximize

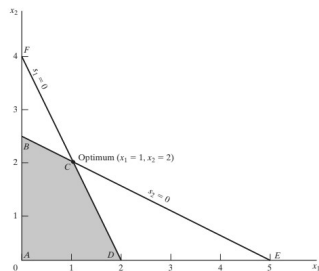
$$z = 2x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \leq 4$$

$$x_1 + 2x_2 \leq 5$$

$$x_1, x_2 \geq 0$$



Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	Feasible?	Objective value, z
(x_1, x_2)	(s_1, s_2)	(4, 5)	A	Yes	0
(x_1, s_1)	(x_2, s_2)	(4, -3)	F	No	-
(x_1, s_2)	(x_2, s_1)	(2.5, 1.5)	B	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2, 3)	D	Yes	4
(x_2, s_2)	(x_1, s_1)	(5, -6)	E	No	-
(s_1, s_2)	(x_1, x_2)	(1, 2)	C	Yes	8 (optimum)

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ITERATIVE NATURE OF THE SIMPLEX METHOD

The simplex method moves iteratively from one corner point to an adjacent corner point that improves the objective value.

- 1 **Start:** Always starts at the origin where decision variables are zero.
- 2 **Direction:** Increase the variable with the largest rate of improvement in z .
- 3 **Path:** Moves alongside the edges of the solution space.

The simplex method examines only basic feasible solutions, not all possible ones.

INITIAL SIMPLEX TABLEAU: REDDY MIKKS MODEL

Objective Equation: $z - 5x_1 - 4x_2 = 0$

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Sol
z	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

Current Solution: $x_1 = 0, x_2 = 0$ (Nonbasic). $s_1 = 24, s_2 = 6, s_3 = 1, s_4 = 2$ (Basic).

STEP 1: OPTIMALITY CONDITION

Rule (Maximization Problem)

If all coefficients of nonbasic variables in the z-row are **nonnegative**, the current solution is optimal.

Otherwise: Choose the nonbasic variable with the **most negative coefficient** in the z-row as the **entering variable**.

From the initial tableau:

- Coefficient of $x_1 = -5$
- Coefficient of $x_2 = -4$

Decision: x_1 enters the basis.

STEP 2: FEASIBILITY CONDITION (RATIO TEST)

Goal: Determine how much the entering variable can increase while maintaining feasibility.

- Each ratio indicates when a constraint becomes binding.
- The **minimum nonnegative ratio** identifies the first constraint to be violated.
- This ensures the next basic solution remains feasible.

Ratio Test

$$\text{Ratio} = \frac{\text{Solution value}}{\text{Positive coefficient in entering column}}$$

Ignore zero or negative denominators, since they do not restrict an increase in the entering variable.

- $s_1: 24/6 = 4$ (**min**)
- $s_2: 6/1 = 6$
- $s_3: 1/(-1)$
(ignored)
- $s_4: 2/0$ (ignored)

s_1 leaves the basis

STEP 3: PIVOT OPERATION (GAUSS–JORDAN)

Goal: Replace the leaving variable with the entering variable while preserving feasibility.

Pivot Element: Entry at the intersection of the **entering column** and **leaving row**.

Gauss–Jordan Operations

① **Normalize Pivot Row:**

$$\text{New pivot row} = \frac{\text{Current pivot row}}{\text{Pivot element}}$$

② **Eliminate Other Entries (including z):**

$$\text{New row} = \text{Current row} - (\text{column coefficient}) \times (\text{new pivot row})$$

This transforms the entering variable into a basic variable.

ITERATION 1: PIVOT CALCULATIONS

Pivot Element: 6 (row s_1 , column x_1)

1. New Pivot Row (x_1 row):

$$s_1/6 \Rightarrow (0, 1, 2/3, 1/6, 0, 0, 0, 4)$$

2. Update z-row:

$$\begin{aligned} Z_{\text{new}} &= Z_{\text{old}} - (-5) \times (\text{new } x_1 \text{ row}) \\ &= (1, -5, -4, 0, \dots) + 5 \times (0, 1, 2/3, \dots) \\ &= (1, 0, -2/3, 5/6, 0, 0, 0, 20). \end{aligned}$$

The same operation is applied to all remaining rows.

TABLEAU AFTER ITERATION 1

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Sol
z	1	0	$-2/3$	$5/6$	0	0	0	20
x_1	0	1	$2/3$	$1/6$	0	0	0	4
s_2	0	0	$4/3$	$-1/6$	1	0	0	2
s_3	0	0	$5/3$	$1/6$	0	1	0	5
s_4	0	0	1	0	0	0	1	2

Optimality Check: The z -row still has a negative coefficient ($-2/3$ under x_2).

Conclusion: The current solution is **not optimal**.

ITERATION 2: SELECTING PIVOT

Entering Variable: x_2 (most negative coefficient in z-row)

Ratio Test:

- $x_1: 4/(2/3) = 6$
- $s_2: 2/(4/3) = 1.5$ **(minimum)**
- $s_3: 5/(5/3) = 3$
- $s_4: 2/1 = 2$

Decision: s_2 leaves the basis.

FINAL OPTIMAL TABLEAU

After performing Gauss-Jordan operations again:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Sol
z	1	0	0	3/4	1/2	0	0	21
x_1	0	1	0	0	0	3
x_2	0	0	1	0	0	1.5
s_3	0	0	0	1	0	2.5
s_4	0	0	0	0	1	0.5

Termination Condition: All coefficients in the z -row are nonnegative.

Optimal Solution:

- Decision Variables: $x_1 = 3$, $x_2 = 1.5$
- Objective Value: $z = 21$

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RESOURCE STATUS FROM THE FINAL TABLEAU

The status of resources is determined by the slack variables (s_1, s_2, s_3, s_4).

- **Scarce Resource:** Slack variable is zero (activity used all resources).
 - ▶ $s_1 = 0 \rightarrow$ Raw Material M1 is scarce.
 - ▶ $s_2 = 0 \rightarrow$ Raw Material M2 is scarce.
- **Abundant Resource:** Slack variable is positive.
 - ▶ $s_3 = 2.5 \rightarrow$ Market Limit is abundant.
 - ▶ $s_4 = 0.5 \rightarrow$ Demand Limit is abundant.

OPTIMALITY CONDITION: SUMMARY

Maximization Problem

- **Entering variable:** Most negative coefficient in the z-row
- **Stop when:** All z-row coefficients ≥ 0

Minimization Problem

- **Entering variable:** Most positive coefficient in the z-row
- **Stop when:** All z-row coefficients ≤ 0

These conditions ensure that no further improvement in the objective value is possible.

FEASIBILITY AND PIVOTING: SUMMARY

These rules apply to both maximization and minimization problems.

Feasibility Condition (Leaving Variable)

Select the basic variable with the **smallest nonnegative ratio**

$$\frac{\text{Solution value}}{\text{Entering column coefficient}}$$

Only strictly positive denominators are considered.

Gauss-Jordan Operations

1. New Pivot Row = Current Pivot Row / Pivot Element
2. New Row = Current Row - (Column Coeff \times New Pivot Row)

KEY TAKEAWAYS

- The feasible region of an LPP is a convex polytope; the corner-point theorem guarantees the optimum is always at a vertex
- Standard form: $\min \mathbf{c}^T \mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ (all constraints as equalities via slack variables)
- A Basic Feasible Solution (BFS) sets $n - m$ non-basic variables to zero; the Simplex Method walks between adjacent BFS vertices
- Pivot rule: entering variable has most negative reduced cost \bar{c}_j ; leaving variable determined by the minimum ratio test

Thank you :)