Optimization Techniques for Machine Learning

AMLZC326 · #02 Spectral theorem

Anshid Aboobacker

Motivation

- Every optimisation algorithm eventually asks: *"Is this Hessian positive definite?"* The answer lies in eigenvalues.
- This lecture builds the algebraic machinery — determinants, eigenvalues, matrix rank — leading to the Spectral Theorem.
- The Spectral Theorem is the foundation for PCA, SVD, and second-order optimality conditions.

Learning Objectives

By the end of this lecture you should be able to:

- Compute determinants and understand their geometric meaning (signed volume)
- Find eigenvalues and eigenvectors via the characteristic polynomial
- Determine matrix rank using row echelon form or the rank-nullity theorem
- State and apply the Spectral Theorem: every real symmetric matrix diagonalises orthogonally

Table of Contents

DETERMINANTS

- **Definition:** The determinant is a function mapping a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ to a real number, denoted $\det(\boldsymbol{A})$ or $|\boldsymbol{A}|$.
- For 2×2 matrices: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$
- Eg: $\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = (3)(2) - (1)(1) = 5.$
- For a 3×3 matrix (expansion along first row):
$$\det(\boldsymbol{A}) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
- Eg:
$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}, \det(\boldsymbol{A}) = (1)(2) - (0)(3) + (1)(2) = 4.$$

DETERMINANT OF $n \times n$ MATRICES

Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. Then, for each $j = 1, \ldots, n$:

Expansion along column j: $\det(\boldsymbol{A}) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det(\boldsymbol{A}_{k,j})$

Expansion along row j: $\det(\boldsymbol{A}) = \sum_{k=1}^{n} (-1)^{k+j} a_{jk} \det(\boldsymbol{A}_{j,k})$

- $\boldsymbol{A}_{k,j} \in \mathbb{R}^{(n-1)\times(n-1)}$ denotes the submatrix obtained by deleting row k and column j of $\boldsymbol{A}$.
- $\det(\boldsymbol{A}_{k,j})$ is called the **minor** corresponding to a_{kj}.
- $(-1)^{k+j} \det(\boldsymbol{A}_{k,j})$ is called the **cofactor** of a_{kj}.

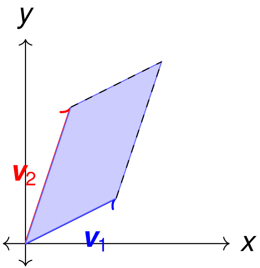
Determinant: Geometric Interpretation

- A matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ defines a linear transformation $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$
- The determinant measures how this transformation scales
  - area in $\mathbb{R}^2$,
  - volume in $\mathbb{R}^3$,
  - n-dimensional volume in $\mathbb{R}^n$.
- Determinant represents the signed volume of the parallelepiped formed by the column vectors of $\boldsymbol{A}$ with respect to the standard ordered basis.
  - $|\det(\boldsymbol{A})|$ = scaling factor of area / volume
  - $\det(\boldsymbol{A}) > 0$: orientation preserved (no reflection)
  - $\det(\boldsymbol{A}) < 0$: orientation reversed (reflection)
  - $\det(\boldsymbol{A}) = 0$: dimension collapses (information lost)

EXAMPLE IN $\mathbb{R}^2$

Let

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = [\, \boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \,], \quad \boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



- $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ form a parallelogram in $\mathbb{R}^2$.
- Area: $\det(\boldsymbol{A}) = (2)(3) - (1)(1) = 5$
- Interpretation:
  - Area scaled by factor 5
  - Orientation preserved
  - $\boldsymbol{A}$ is invertible

If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent (e.g. $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$), the parallelogram collapses to a line and $\det(\boldsymbol{A}) = 0$.

Properties of the Determinant

- $\boldsymbol{A}$ is invertible $\iff$ $\det(\boldsymbol{A}) \neq 0$ $\iff$ $\mathrm{rank}(\boldsymbol{A}) = n$.
- Determinant and matrix operations:
  - $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$.
  - $\det(\boldsymbol{A}^\top) = \det(\boldsymbol{A})$.
  - If $\boldsymbol{A}$ is invertible, then $\det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A})$.
  - Similar matrices have the same determinant.
- Effect of row/column operations:
  - Adding a multiple of one row (or column) to another does not change $\det(\boldsymbol{A})$.
  - Multiplying a row (or column) by λ scales $\det(\boldsymbol{A})$ by λ.
  - Swapping two rows (or columns) changes the sign of $\det(\boldsymbol{A})$.

The Trace

- **Definition:** The trace of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is the sum of its diagonal entries:

$$\mathrm{tr}(\boldsymbol{A}) := \sum_{i=1}^{n} a_{ii}$$

- **Properties:**
  - Linearity: $\mathrm{tr}(\boldsymbol{A} + \boldsymbol{B}) = \mathrm{tr}(\boldsymbol{A}) + \mathrm{tr}(\boldsymbol{B})$.
  - Cyclic Permutation invariance: $\mathrm{tr}(\boldsymbol{ABC}) = \mathrm{tr}(\boldsymbol{BCA}) = \mathrm{tr}(\boldsymbol{CAB})$.
  - Basis independence: The trace of a linear mapping is independent of the choice of basis. If $\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{AS}$, then $\mathrm{tr}(\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{S}^{-1}\boldsymbol{AS}) = \mathrm{tr}(\boldsymbol{ASS}^{-1}) = \mathrm{tr}(\boldsymbol{A})$.

Characteristic Polynomial

- To find eigenvalues of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, define the **characteristic polynomial**

$$p_{\boldsymbol{A}}(\lambda) := \det(\boldsymbol{A} - \lambda \boldsymbol{I}), \quad \lambda \in \mathbb{R}.$$

- Expanding gives a polynomial of degree n:

$$p_{\boldsymbol{A}}(\lambda) = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0.$$

- Link to determinant and trace:
  - Constant term: $c_0 = \det(\boldsymbol{A})$.
  - Coefficient of λ^{n-1}: $c_{n-1} = (-1)^{n-1} \mathrm{tr}(\boldsymbol{A})$.

Table of Contents

Eigenvalues and Eigenvectors

- Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a square matrix. Then $\lambda \in \mathbb{R}$ is an **eigenvalue** of $\boldsymbol{A}$ and $\boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}$ is the corresponding **eigenvector** of $\boldsymbol{A}$ if:
$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$
- Intuition:
  - The linear mapping $\boldsymbol{A}$ stretches the vector $\boldsymbol{x}$ by a factor λ.
  - The direction of $\boldsymbol{x}$ is preserved (codirected) or flipped.
- Finding Eigenvalues: λ is an eigenvalue of matrix $\boldsymbol{A}$ iff it is a root of the characteristic polynomial: $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$
- If $\boldsymbol{x}$ is an eigenvector associated with λ, then for any $c \in \mathbb{R} \setminus \{0\}$, $c\boldsymbol{x}$ is also an eigenvector: $\boldsymbol{A}(c\boldsymbol{x}) = c(\boldsymbol{A}\boldsymbol{x}) = c(\lambda \boldsymbol{x}) = \lambda(c\boldsymbol{x})$

EIGENSPACES AND MULTIPLICITIES

- **Eigenspace** $E_\lambda := \ker(\boldsymbol{A} - \lambda \boldsymbol{I})$,
  the set of all eigenvectors corresponding to the eigenvalue λ, together with $\mathbf{0}$.
- **Spectrum/Eigenspectrum:** the set of all eigenvalues of $\boldsymbol{A}$.
- **Multiplicities:**
  - **Algebraic multiplicity:** the number of times λ appears as a root of the characteristic polynomial $p_{\boldsymbol{A}}(\lambda)$.
  - **Geometric multiplicity:** the dimension of the eigenspace E_λ.
  - For every eigenvalue λ: $1 \leq$ geom. mult. $\leq$ alg. mult.
- Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with algebraic multiplicity), then,

$$\det(\boldsymbol{A}) = \prod_{i=1}^{n} \lambda_i, \qquad \mathrm{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} \lambda_i.$$

Key Properties

- $\boldsymbol{A}$ and $\boldsymbol{A}^\top$ possess the same eigenvalues, but not necessarily the same eigenvectors.
- Similar matrices (representing the same linear map in different bases) have the same eigenvalues.
  - Eigenvalues, determinant, and trace are basis-invariant.
- Eigenvectors associated with **distinct** eigenvalues are linearly independent.
- The following statements are equivalent for $\lambda \in \mathbb{R}$:
  1. λ is an eigenvalue of $\boldsymbol{A}$.
  2. There exists $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$.
  3. The system $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \mathbf{0}$ has a non-trivial solution.
  4. $\mathrm{rank}(\boldsymbol{A} - \lambda \boldsymbol{I}) < n$.
  5. $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$.

EXAMPLES

Example 1 The Identity matrix $\boldsymbol{I} \in \mathbb{R}^{n \times n}$

1. Polynomial: $p_{\boldsymbol{I}}(\lambda) = \det(\boldsymbol{I} - \lambda \boldsymbol{I}) = (1 - \lambda)^n = 0$.
2. Eigenvalue $\lambda = 1$ with algebraic multiplicity n.
3. Eigenspace is all of $\mathbb{R}^n$ (any non-zero vector is an eigenvector).

Example 2 Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.

1. Characteristic polynomial:
$$p_{\boldsymbol{A}}(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det\left(\begin{bmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{bmatrix}\right)$$
$$= (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10$$

Examples

2 Eigenvalues: $\lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5) = 0$
The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$.

3 Eigenspaces: Solve $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = 0$

For $\lambda_2 = 5$: $\begin{bmatrix} 4-5 & 2 \\ 1 & 3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$

Solution: $x_1 = 2x_2$. Basis vector: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $E_5 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

For $\lambda_1 = 2$: Analogous calculation yields $x_1 = -x_2$.
$E_2 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

EXAMPLES

Example 3 Consider the shear matrix $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

- Eigenvalues: $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = (2 - \lambda)^2 = 0$.
- $\lambda = 2$ is a repeated root.
- Algebraic Multiplicity: 2.

Solving $(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{0} \implies x_2 = 0$

- Only one independent eigenvector: $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- Geometric Multiplicity: 1.

Table of Contents

Computing the Rank of a Matrix

What is Rank?

- Rank measures the number of **independent rows or columns**.

Method 1: Row Echelon Form (REF / RREF)

- **Idea:** Use Gaussian elimination to simplify the matrix.
- Reduce the matrix to **Row Echelon Form (REF)** or **Reduced REF**.
- **Rule:** rank(A) = number of non-zero rows in REF
- Equivalently, rank = number of pivot positions.

Computing the Rank of a Matrix

Example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{R_2 - 2R_1,\ R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Number of non-zero rows = 2
- Hence, $\text{rank}(\boldsymbol{A}) = 2$

Computing the Rank of a Matrix

Method 2: Determinants of submatrices (Minors)

- **Idea:** Rank depends on the largest square submatrix with non-zero determinant.
- Rank = **maximum order of a non-vanishing minor**.

Procedure for $\boldsymbol{A} \in \mathbb{R}^{m \times n}$:

1. Set $k = \min(m, n)$.
2. Check all $k \times k$ submatrices.
3. If any determinant $\neq 0$, then $\text{rank}(\boldsymbol{A}) = k$.
4. Otherwise, reduce k by one and repeat.

Computing the Rank of a Matrix

Example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$$

- $\det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0$
- $\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = -2 \neq 0$
- Therefore, $\text{rank}(\boldsymbol{A}) = 2$

Computing the Rank of a Matrix

Method 3: Rank–Nullity Theorem

- For any matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$:

$$\operatorname{rank}(\boldsymbol{A}) + \operatorname{nullity}(\boldsymbol{A}) = n$$

- **Nullity** = dimension of the solution space of $\boldsymbol{Ax} = \mathbf{0}$.
- Useful when solving $\boldsymbol{Ax} = \mathbf{0}$ is easier than row reduction.

Computing the Rank of a Matrix

Example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \quad (2 \times 3,\ n = 3)$$

- $\boldsymbol{Ax} = \mathbf{0} \Rightarrow x_1 + x_2 + x_3 = 0$
- General solution:

$$\boldsymbol{x} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- $\text{nullity}(\boldsymbol{A}) = 2$
- $\text{rank}(\boldsymbol{A}) = n - \text{nullity} = 3 - 2 = 1$

Table of Contents

SYMMETRIC MATRICES

Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ with $\boldsymbol{x} \neq \boldsymbol{0}$.

Symmetric matrices ($\boldsymbol{A}^\top = \boldsymbol{A}$):

$$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^\top(\lambda \boldsymbol{x}) = \lambda\, \boldsymbol{x}^\top \boldsymbol{x}.$$

$$\lambda = \frac{\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}}$$

Since $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \in \mathbb{R}$ and $\boldsymbol{x}^\top \boldsymbol{x} > 0$, we conclude $\lambda \in \mathbb{R}$.

- All eigenvalues are real.
- Eigenvectors corresponding to distinct eigenvalues are orthogonal.

SKEW-SYMMETRIC & ORTHOGONAL MATRICES

- **Skew-symmetric matrices ($\boldsymbol{A}^\top = -\boldsymbol{A}$):** using the complex inner product, $\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{x})^* = \boldsymbol{x}^*\boldsymbol{A}^\top\boldsymbol{x} = -\,\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{x}$
  Hence $\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{x}$ is purely imaginary or zero, and

$$\lambda = 0 \text{ or } \lambda = \frac{\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^*\boldsymbol{x}} \in i\mathbb{R}.$$

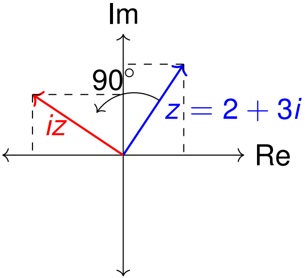
- **Orthogonal matrices ($\boldsymbol{A}^\top\boldsymbol{A} = \boldsymbol{I}$):**

$$\|\boldsymbol{A}\boldsymbol{x}\|^2 = (\boldsymbol{A}\boldsymbol{x})^\top(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{x}^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^\top\boldsymbol{x}.$$

  If $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, then $|\lambda|^2\|\boldsymbol{x}\|^2 = \|\boldsymbol{x}\|^2$, which implies $|\lambda| = 1$.

Complex Numbers

- A **complex number** is of the form $z = a + ib$, with $a, b \in \mathbb{R}$, $\quad i^2 = -1$
- Real part: $\Re(z) = a$
- Imaginary part: $\Im(z) = b$
- $(a+ib)+(c+id) = (a+c)+i(b+d)$
- $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$
- **Geometric interpretation:** multiplication by i corresponds to a rotation by 90° in the complex plane (e.g. $z \mapsto iz$).



Complex Numbers

- Complex conjugate: $\bar{z} = a - ib$
  $\bar{z}$ reflects z across the real axis.
- Magnitude: $|z| = \sqrt{a^2 + b^2}$
  $|z|$ is the distance from the origin.
- Why complex numbers appear:
  - Not all real polynomials have real roots.
  - Fundamental Theorem of Algebra:
    Every polynomial has roots in $\mathbb{C}$.
- Relevance to eigenvalues: real matrices can have complex eigenvalues, which always occur in conjugate pairs.

Special Matrices (Complex Case)

- **Hermitian matrix** (complex analogue of symmetric): $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, $\boldsymbol{A}^* = \boldsymbol{A}$
- **Skew-Hermitian matrix**(complex analogue of skew-symmetric): $\boldsymbol{A}^* = -\boldsymbol{A}$
- **Unitary matrix** (complex analogue of orthogonal): $\boldsymbol{A}^*\boldsymbol{A} = \boldsymbol{I}$
- **Eigenvalue behavior (same proofs as before):**
  - Hermitian $\Rightarrow$ eigenvalues are real.
  - Skew-Hermitian $\Rightarrow$ eigenvalues are purely imaginary or zero.
  - Unitary $\Rightarrow$ eigenvalues satisfy $|\lambda| = 1$.

Table of Contents

The Spectral Theorem

If $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is symmetric, there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $\boldsymbol{A}$, and each eigenvalue is real.

Implications:

- A symmetric matrix can always be diagonalized: $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^\top$
- $\boldsymbol{P}$ is an orthogonal matrix (columns are orthonormal eigenvectors).
- $\boldsymbol{D}$ is a diagonal matrix containing the eigenvalues.
- This decomposition exists even if eigenvalues are repeated (non-defective).

Example: Symmetric Matrix with Repeated Eigenvalues

Consider

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

- $p_{\boldsymbol{A}}(\lambda) = -(\lambda - 1)^2(\lambda - 7)$.
- Eigenvalues: $\lambda = 7$, $\lambda = 1$ (algebraic multiplicity 2)

Eigenspaces:

- $E_7 = \mathrm{span}\{[1, 1, 1]^\top\}$.
- $E_1 = \mathrm{span}\{\boldsymbol{x}_1 = [-1, 1, 0]^\top,\ \boldsymbol{x}_2 = [-1, 0, 1]^\top\}$.

Note: $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are linearly independent but not orthogonal.

Any linear combination of eigenvectors for λ is also an eigenvector for λ. Applying Gram–Schmidt to $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ gives:

$\boldsymbol{x}_1' = \left[-1, 1, 0\right]^T, \qquad \boldsymbol{x}_2' = \frac{1}{2}\left[-1, -1, 2\right]^T$

Now we have an orthogonal eigenbasis for the entire space $\mathbb{R}^3$.

Key Takeaways

- The determinant measures signed volume; $\det(A) = 0 \Leftrightarrow A$ is singular (non-invertible)
- Eigenvalues λ and eigenvectors $\boldsymbol{x}$ satisfy $A\boldsymbol{x} = \lambda\boldsymbol{x}$; they reveal the intrinsic scaling directions of a linear map
- The Spectral Theorem: every real symmetric A satisfies $A = Q\Lambda Q^\top$ with Q orthogonal and Λ real diagonal
- Positive definiteness $\Leftrightarrow$ all $\lambda_i > 0$ — the cornerstone of second-order optimality conditions throughout the course

Thank you :)