University of Calicut

B.Sc. Mathematics Honours — CUFYUGP 2024

Semester I (Minor) — MAT1MN104

Mathematical Logic, Set Theory and Combinatorics

A Student Study Guide

Build intuition. Understand deeply. Prepare for exams.

Based on:

Thomas Koshy,
Discrete Mathematics with Applications, 1st Ed., Academic Press, 2003

Academic Year 2026–27

Preface

This book is a study guide for **MAT1MN104 — Mathematical Logic, Set Theory and Combinatorics**, a Minor course in the B.Sc. Mathematics Honours programme at colleges affiliated with the University of Calicut under the CUFYUGP 2024 regulations.

Who is this guide for? You are a first-year Mathematics Honours student meeting formal logic and rigorous set theory for the first time. No background in proof-writing is assumed. We build every idea from everyday reasoning first, then attach the precise mathematical language and symbols on top of it.

How to read this guide. Each chapter covers one module of the syllabus. Within each chapter, topics follow a consistent flow:

1. **Intuition box** (green) — the big idea in plain language, before any symbols.
2. **Definition box** (blue) — the precise mathematical statement, in Koshy's notation.
3. **Worked Example box** (orange) — a fully solved example with every step shown. This guide leans heavily on worked examples.
4. **Exam Tip box** (red) — what the university exam specifically tests on this topic, and what is exempted.
5. **Key Takeaway box** (purple) — the one sentence to carry forward.
6. **Warning box** (yellow) — a common mistake to avoid.
7. **Module V Extension box** (slate) — an open-ended topic from Module V of the syllabus, folded into the chapter it naturally extends. These are enrichment only and are *never* externally examined.

Not every topic uses all seven elements; short topics may only have a definition and an example.

A note on proofs. The syllabus explicitly exempts *all* proofs from the end-semester examination (see the syllabus note: "Proofs of all the results are also exempted for the end semester exam"). This guide therefore keeps proofs light: most results come with a short, intuitive justification rather than a formal derivation. Where the textbook's proof is genuinely illuminating, a compact version is included inline, but it is never the focus. Your energy is best spent on definitions, notation, and worked examples — that is what the exam tests.

Exercises. Each chapter ends with a set of practice problems arranged in *increasing order of difficulty*: the earlier problems are direct applications of the chapter's definitions and formulas, and the later ones combine several ideas or require a bit more thought. There are no explicit difficulty labels; treat the ordering itself as the guide. Since proofs are not examined, exercises focus on computation, symbolic translation, and applying definitions correctly.

Previous year question paper. The first batch of CUFYUGP 2024 students sat the MAT1MN104 external exam in October 2024. That paper is reproduced in the `previous-year-questions` folder alongside this guide. Throughout the main text, worked examples drawn from or closely modelled on that paper are tagged [Oct 2024].

Exam pattern.

Section	Marks per question	Maximum marks
A (short answer)	3	Ceiling 24
B (medium answer)	6	Ceiling 36
C (long answer)	10	Any one of two options
External total		70
Internal		30

Every module (I–IV) is guaranteed a minimum of 15 external marks, so no module can safely be skipped.

Notation. This course follows **Thomas Koshy's notation throughout**, confirmed directly from both the textbook and the actual October 2024 question paper. The most important departures from other textbooks you may have seen:

- Negation is written $\sim p$ (a tilde), *not* $\neg p$.
- Set complement is written A' (a prime), *not* A^c.
- Implication is $p \to q$; biconditional is $p \leftrightarrow q$; conjunction $\wedge$; disjunction $\vee$.
- r-permutations of n elements: $P(n,r) = \dfrac{n!}{(n-r)!}$. r-combinations of n elements: $C(n,r) = \dfrac{n!}{r!(n-r)!}$.
- A function is written $f : X \to Y$, with domain $\mathrm{Dom}(f)$ and codomain $\mathrm{codom}(f)$.
- Cardinality (size) of a set A is $|A|$.

Keep this page in mind if you ever consult a different textbook (Rosen, Epp, ...) — the mathematics is the same, but a few symbols differ, and *this course's exam expects Koshy's symbols*.

Contents

Preface i

1 Mathematical Logic 1
- 1.1 Propositions . . . 1
  - 1.1.1 Conjunction and Disjunction . . . 2
  - 1.1.2 The Conditional Statement . . . 3
  - 1.1.3 The Biconditional . . . 4
  - 1.1.4 Order of Precedence . . . 4
  - 1.1.5 Tautology, Contradiction, and Contingency . . . 5
- 1.2 Logical Equivalences . . . 5
  - 1.2.1 The Laws of Logic . . . 6
- 1.3 Quantifiers . . . 7
- 1.4 Arguments . . . 8
- Chapter Summary . . . 10
- Practice Problems . . . 10

2 Set Theory 11
- 2.1 The Concept of a Set . . . 11
  - 2.1.1 Subsets, Equal Sets, Power Sets . . . 12
  - 2.1.2 Finite and Infinite Sets . . . 13
- 2.2 Operations with Sets . . . 14
  - 2.2.1 The Laws of Sets . . . 15
  - 2.2.2 The Cartesian Product . . . 16
- 2.3 The Cardinality of a Set . . . 17
- Module V Extension — Relations and Digraphs . . . 18
- Chapter Summary . . . 19
- Practice Problems . . . 19

3 Functions and Matrices 20
- 3.1 The Concept of a Function . . . 20
  - 3.1.1 Piecewise Functions, and New Functions from Old . . . 21
- 3.2 Special Functions . . . 22
  - 3.2.1 Floor and Ceiling Functions . . . 22
  - 3.2.2 Characteristic Function . . . 23
  - 3.2.3 Mod and Div Functions . . . 23
- 3.3 Matrices . . . 24
  - 3.3.1 Matrix Addition and Scalar Multiplication . . . 25
  - 3.3.2 Matrix Multiplication . . . 25
- Module V Extension — Basic Calculus Concepts . . . 26

Module V Extension — Root-Finding Methods . . . 27
Module V Extension — Gauss-Jordan Method . . . 27
Chapter Summary . . . 28
Practice Problems . . . 28

4 Combinatorics and Discrete Probability **30**
4.1 The Fundamental Counting Principles . . . 30
4.2 Permutations . . . 31
4.2.1 Cyclic Permutations . . . 32
4.3 Combinations . . . 33
4.4 Discrete Probability . . . 34
4.4.1 Mutually Exclusive Events . . . 36
Module V Extension — Conditional Probability and Beyond . . . 37
Chapter Summary . . . 38
Practice Problems . . . 38

Quick Reference **39**

Hints and Answers **40**

Chapter 1

Mathematical Logic

What you will learn in this chapter.

- How to turn everyday statements into precise, symbolic propositions.
- The five logical connectives — conjunction, disjunction, conditional, biconditional, negation — and their truth tables.
- The converse, inverse, and contrapositive of a conditional statement.
- How to tell whether two logical expressions mean the same thing (logical equivalence), and the standard laws of logic.
- Quantifiers: how to say "for all" and "there exists" symbolically.
- How to decide whether an everyday argument is logically valid.

1.1 Propositions

Intuition

Mathematics needs sentences that are unambiguously true or false, with no room for opinion. "Kerala is beautiful" is not such a sentence — people disagree. "7 is a prime number" is such a sentence. Logic is the study of exactly these black-and-white sentences, and the ways we can combine them.

Definition

A **proposition** (or **statement**) is a declarative sentence that is either **true** or **false**, but not both. We write $t(p)$ for the **truth value** of a proposition p: $t(p) = T$ if p is true, and $t(p) = F$ if p is false.

Worked Example

- "$2 + 2 = 4$" is a proposition (true).
- "The moon is made of cheese" is a proposition (false).
- "$x + 1 = 5$" is *not* a proposition on its own — its truth depends on the unknown value x. (Once x is fixed, it becomes a proposition.)
- "Please close the door" and "What time is it?" are not propositions — they are not declarative sentences.

1.1.1 Conjunction and Disjunction

Definition

Let p and q be propositions.

- The **conjunction** of p and q, written $p \wedge q$ (read "p and q"), is true exactly when *both* p and q are true.
- The **disjunction** of p and q, written $p \vee q$ (read "p or q"), is true exactly when *at least one* of p, q is true. (This is the *inclusive* or.)
- The **negation** of p, written $\sim p$ (read "not p"), is true exactly when p is false.

Warning

Koshy's book (and this course) writes negation as $\sim p$, a **tilde**. Many other textbooks use $\neg p$ or $\bar{p}$ instead — these all mean exactly the same thing, but on your exam paper, write $\sim p$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	$\sim p$
T	F
F	T

Worked Example

Let p: "Kochi is in Kerala" (true) and q: "5 is even" (false).

- $p \wedge q$: "Kochi is in Kerala and 5 is even" — **false** (since q is false).
- $p \vee q$: "Kochi is in Kerala or 5 is even" — **true** (since p is true).
- $\sim q$: "5 is not even" — **true**.

Worked Example — Building a Truth Table

Construct the truth table for $\sim p \vee q$.

Solution. There are two propositional variables, so there are $2^2 = 4$ rows.

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

In general, a truth table for an expression with n propositional variables has 2^n rows.

1.1.2 The Conditional Statement

Definition

The **conditional** (or **implication**) $p \to q$ (read "if p then q", or "p implies q") is false only when p is true and q is false; it is true in every other case. Here p is the **hypothesis** (or premise) and q is the **conclusion**.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Intuition

Think of $p \to q$ as a promise: "If you wax my car (p), I will pay you \$25 (q)." The promise is broken *only* if you wax the car and I don't pay — that is the one false row. If you never wax the car, I haven't broken any promise no matter what I do; so when p is false, the implication is automatically (*vacuously*) true.

Warning

In everyday English, $p \to q$ usually suggests p *causes* q. In logic there is no such requirement. "If the power is on, then $3 + 5 = 8$" is a *true* implication (the conclusion is true, full stop), even though the two halves have nothing to do with each other.

Converse, Inverse, and Contrapositive

Definition

From an implication $p \to q$ we build three related implications:

- **Converse**: $q \to p$ (swap hypothesis and conclusion).
- **Inverse**: $\sim p \to \sim q$ (negate both).
- **Contrapositive**: $\sim q \to \sim p$ (negate both, *and* swap).

Worked Example

Let $p \to q$ be: "If $\triangle ABC$ is equilateral, then it is isosceles."

- Converse ($q \to p$): If $\triangle ABC$ is isosceles, then it is equilateral. *(False in general!)*
- Inverse ($\sim p \to \sim q$): If $\triangle ABC$ is not equilateral, then it is not isosceles. *(False in general!)*
- Contrapositive ($\sim q \to \sim p$): If $\triangle ABC$ is not isosceles, then it is not equilateral. *(True.)*

Key Takeaway

A conditional and its **contrapositive** always have the same truth value. A conditional and its **converse** (equivalently, its inverse) generally do *not*. This is one of the most exam-relevant facts in the whole chapter — see Section 1.2.

1.1.3 The Biconditional

Definition

The **biconditional** $p \leftrightarrow q$ (read "p if and only if q") is true exactly when p and q have the *same* truth value.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Worked Example

Let S denote the sum of the digits of 2034 ($S = 2 + 0 + 3 + 4 = 9$). "2034 is divisible by 3 if and only if S is divisible by 3" is a **true** biconditional: both halves (2034 divisible by 3, and 9 divisible by 3) happen to be true together.

1.1.4 Order of Precedence

Definition

When a logical expression has no parentheses, evaluate the operators from **highest to lowest** precedence:

$$(1)\ \sim \quad (2)\ \wedge \quad (3)\ \vee \quad (4)\ \rightarrow \quad (5)\ \leftrightarrow .$$

Operators of equal precedence are evaluated left to right. Parenthesised subexpressions are always evaluated first.

Worked Example

$p \rightarrow q \wedge \sim q \rightarrow \sim p$ is grouped as $\big[(p \rightarrow q) \wedge (\sim q)\big] \rightarrow (\sim p)$, *not* as $p \rightarrow \big[q \wedge \sim q \rightarrow \sim p\big]$, because $\wedge$ binds tighter than $\rightarrow$.

1.1.5 Tautology, Contradiction, and Contingency

Definition

A proposition (built from simpler propositions using connectives) is a

- **tautology** if it is **true** for every combination of truth values of its variables;
- **contradiction** if it is **false** for every combination;
- **contingency** if it is *neither* — true for some combinations and false for others.

Worked Example

- $p \vee \sim p$ is a **tautology**: whatever p is, one of $p, \sim p$ is true.
- $p \wedge \sim p$ is a **contradiction**: it is impossible for p and $\sim p$ to both be true.
- $p \to q$ is a **contingency**: it is true in three rows of its truth table and false in one.

Exam Tip

Section 1.1 (this section) is fully examinable except: the switching-network material and Example 1.16 in the textbook. Expect direct questions such as "give truth tables for conjunction and disjunction" (3 marks) or "explain converse, inverse, and contrapositive with examples" (10 marks) — both appeared on the October 2024 paper [Oct 2024].

1.2 Logical Equivalences

Intuition

Just as $2x$ and $x + x$ are two different-looking expressions with the *same value* for every x, two logical expressions can look different but have identical truth tables. Recognising these "disguises" lets you simplify complicated logical statements — a skill you already used implicitly when you noticed $p \to q \equiv \sim q \to \sim p$ in the last section.

Definition

Two propositions P and Q (each built from the same variables) are **logically equivalent**, written $P \equiv Q$, if they have identical truth tables — that is, $P \leftrightarrow Q$ is a tautology.

Worked Example — Proving an Equivalence

Show that $p \to q \equiv \sim q \to \sim p$ (the conditional is equivalent to its contrapositive).
Solution. Build one truth table with columns for both sides.

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The third and sixth columns are identical, so $p \to q \equiv \sim q \to \sim p$. [Oct 2024]

1.2.1 The Laws of Logic

Definition

Let p, q, r be propositions, t a tautology, and f a contradiction. The following pairs are all logically equivalent:

Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
Identity laws	$p \wedge t \equiv p$	$p \vee f \equiv p$
Inverse laws	$p \wedge (\sim p) \equiv f$	$p \vee (\sim p) \equiv t$
Domination laws	$p \vee t \equiv t$	$p \wedge f \equiv f$
Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Double negation		$\sim (\sim p) \equiv p$
Associative laws	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
Distributive laws		$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
		$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's laws	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
Implication conversion		$p \to q \equiv \sim p \vee q$
Contrapositive law		$p \to q \equiv \sim q \to \sim p$
Reductio ad absurdum		$p \to q \equiv (p \wedge \sim q) \to f$

Warning

Parentheses are *essential* in the distributive laws. Without them, $p \wedge q \vee r$ is unambiguous (it means $(p \wedge q) \vee r$, since $\wedge$ binds tighter), but that is *not* the same proposition as $p \wedge (q \vee r)$ in general. Do not drop the parentheses when applying a distributive law.

Worked Example — Verifying De Morgan's Law

Verify that $\sim (p \wedge q) \equiv \sim p \vee \sim q$.
Solution. [Oct 2024]

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Columns 4 and 7 match, so $\sim (p \wedge q) \equiv \sim p \vee \sim q$. The same method verifies

$\sim (p \vee q) \equiv \sim p \wedge \sim q$.

Worked Example — Simplifying with the Laws

Simplify $\sim (p \vee \sim q) \vee (\sim p \wedge \sim q)$.
Solution.

$$\begin{aligned}
\sim (p \vee \sim q) \vee (\sim p \wedge \sim q) &\equiv (\sim p \wedge \sim (\sim q)) \vee (\sim p \wedge \sim q) && \text{De Morgan}\\
&\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) && \text{double negation}\\
&\equiv \sim p \wedge (q \vee \sim q) && \text{distributive}\\
&\equiv \sim p \wedge t && \text{inverse law}\\
&\equiv \sim p && \text{identity law}
\end{aligned}$$

Exam Tip

Expect to *apply* named laws in a chain (as above), and to *simplify* a given set expression or logical expression using them (a Section B, 6-mark style question). You will not be asked to prove any of the 20 laws themselves — only to use them.

1.3 Quantifiers

Intuition

"All people are mortal." "Some people have blue eyes." These are propositions too, but they talk about *every* or *some* member of a whole collection, rather than being built from named propositions p, q with connectives. We need two new symbols to write them precisely.

Definition

Let $P(x)$ be a statement about x (a **predicate**).

- The **universal quantifier** $\forall$ (an inverted A, read "for all", "for each", or "for every") asserts $P(x)$ holds for *every* x under consideration: $(\forall x)\, P(x)$.
- The **existential quantifier** $\exists$ (a backward E, read "there exists", "for some", or "for at least one") asserts $P(x)$ holds for *at least one* x: $(\exists x)\, P(x)$.

Worked Example

- "All apples are green": let $P(x)$: "x is green", over the domain of apples. Symbolically, $(\forall x)\, P(x)$.
- "For each integer x, there exists an integer y such that $x+y=0$": $(\forall x)(\exists y)(x+y=0)$. [Oct 2024]
- "There exists an even prime number": $(\exists x)(x \text{ is even} \wedge x \text{ is prime})$.
- "No birds are black": every bird is *not* black, so $(\forall x)(x \text{ is a bird } \rightarrow \sim (x \text{ is black}))$.

Worked Example

Recall subsets from set theory (Chapter 2) are already stated with quantifiers: $A \subseteq B$ means precisely $(\forall x)(x \in A \to x \in B)$, and $A \not\subseteq B$ means $(\exists x)(x \in A \wedge x \notin B)$.

Module V Extension — Not Examined

Negating quantified statements (De Morgan's laws for quantifiers). To negate "for all x, $P(x)$", it suffices to find *one* counterexample: $\sim (\forall x)\, P(x) \equiv (\exists x) \sim P(x)$. Similarly, $\sim (\exists x)\, P(x) \equiv (\forall x) \sim P(x)$. For instance, the negation of "all apples are green" is not "no apples are green" but "there exists an apple that is not green". This mirrors De Morgan's laws for $\wedge$ and $\vee$ from Section 1.2 — $\forall$ behaves like a big $\wedge$ over all cases, and $\exists$ like a big $\vee$. This topic (Koshy, Example 1.29) is marked optional in the syllabus and is not externally examined.

Exam Tip

The syllabus marks Example 1.28 and the quantifier De Morgan's laws as optional/non-examined. What *is* examined: translating an English sentence into quantified symbolic form (and back), as in the examples above — this is a common Section A/B question.

1.4 Arguments

Intuition

An **argument** is a sequence of propositions — the **premises** (or hypotheses) — followed by a **conclusion**, claimed to follow from them. Logic gives us a mechanical way to check whether that claim is actually justified, without needing to trust our gut feeling about the English sentences involved.

Definition

An argument with premises $p_1, p_2, \ldots, p_n$ and conclusion q is **valid** if, whenever *all* the premises are true, the conclusion q must also be true — that is, if

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$$

is a tautology. An argument that is not valid is **invalid** (it contains a logical flaw).

Intuition

To check validity: (1) rewrite each premise and the conclusion symbolically; (2) assume all the premises are true; (3) see whether the laws of logic and the standard inference rules below force the conclusion to be true. If they do, the argument is valid; if you can find truth values making every premise true but the conclusion false, it is invalid.

Definition

The following **inference rules** are themselves tautologies, and can be chained together to validate an argument. In inferential form, the premises are listed above a line and the conclusion below it, preceded by $\therefore$ ("therefore").

Rule	Statement
Conjunction	$p \wedge q \to (p \wedge q)$
Simplification	$p \wedge q \to p$
Addition	$p \to p \vee q$
Law of detachment (*modus ponens*)	$[p \wedge (p \to q)] \to q$
Law of the contrapositive (*modus tollens*)	$[(p \to q) \wedge (\sim q)] \to \sim p$
Disjunctive syllogism	$[(p \vee q) \wedge (\sim p)] \to q$
Hypothetical syllogism	$[(p \to q) \wedge (q \to r)] \to (p \to r)$

Intuition

The **law of detachment** is the workhorse: if you know "p" and you know "if p then q", you may conclude "q". In inferential form:

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore\ q \end{array}$$

Worked Example — Checking Validity

Check the validity of the argument:

> If the computer was down Saturday afternoon, then Mary went to a matinee.
> Either Mary went to a matinee or she took a nap Saturday afternoon.
> Mary did not take a nap that afternoon.
> $\therefore$ The computer was down Saturday afternoon.

Solution. Let p: "the computer was down", q: "Mary went to a matinee", r: "Mary took a nap". The premises are $p \to q$, $q \vee r$, and $\sim r$; the conclusion is p. From $q \vee r$ and $\sim r$, the disjunctive syllogism gives q. But we only have $p \to q$, not $q \to p$ — knowing q does *not* let us conclude p (that would be the fallacy of affirming the converse). So this argument is **invalid**, even though its conclusion happens to sound plausible.

Warning

The single most common invalid-argument trap: from $p \to q$ and q (true), concluding p. This is *not* a rule of logic — it is the **fallacy of affirming the consequent**. Compare it with the genuinely valid law of detachment, which requires p (not q) as the known premise.

Exam Tip

Section 1.4 is examinable in full except Example 1.33. Expect a Section B question giving an argument in English and asking you to check its validity by translating to symbols and applying the inference rules — this uses exactly the same laws-of-logic skills from Section 1.2.

Chapter Summary

- A **proposition** is a declarative sentence with a definite truth value.
- Connectives: $\wedge$ (and), $\vee$ (or), $\sim$ (not, a *tilde*), $\rightarrow$ (implies), $\leftrightarrow$ (iff). Precedence: $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$.
- From $p \rightarrow q$: converse $q \rightarrow p$, inverse $\sim p \rightarrow \sim q$, contrapositive $\sim q \rightarrow \sim p$. The conditional and its contrapositive are always logically equivalent.
- **Tautology** (always T), **contradiction** (always F), **contingency** (neither).
- Two expressions are **logically equivalent** ($\equiv$) if their truth tables match; Table 1.13 (20 laws) is the toolkit for proving/simplifying equivalences, most importantly De Morgan's laws.
- Quantifiers: $\forall$ (for all), $\exists$ (there exists).
- An **argument** is **valid** if the conjunction of its premises implies its conclusion (a tautology); the inference rules, especially the **law of detachment**, are the standard tools for proving validity.

Practice Problems

1. Give truth tables for the conjunction and disjunction of two propositions p and q.
2. Write the negation of: "The exam is easy and the hall is cool."
3. Rewrite symbolically: "For each real number x, there exists a real number y such that $x + y = 0$."
4. Define contradiction, and give an example different from $p \wedge \sim p$.
5. Construct the truth table for $(p \rightarrow q) \wedge (q \rightarrow p)$. Which biconditional is this equivalent to?
6. State the converse, inverse, and contrapositive of: "If a number is divisible by 6, then it is divisible by 3." Which of the three is guaranteed to be true, given the original statement is true?
7. Simplify $(p \wedge q) \vee (p \wedge \sim q)$ using the laws of logic, showing each law used.
8. Verify, using a truth table, that $\sim (p \vee q) \equiv \sim p \wedge \sim q$.
9. Translate into symbols: "Every student who studies passes the exam, but some student who studies did not pass." Is this pair of statements consistent?
10. Check the validity of the argument: "If it rains, the match is cancelled. It did not rain. Therefore, the match was not cancelled." Name the fallacy, if any.
11. Using the laws of logic (not a truth table), show that $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$.
12. Three friends make statements about who ate the last mango: Asha says "Bala did it." Bala says "Chithra did it." Chithra says "Bala is lying." Exactly one of the three is telling the truth. Using propositional logic, determine who ate the mango.

Chapter 2

Set Theory

What you will learn in this chapter.

- How to describe a set by listing its elements or by a defining property.
- Subsets, equal sets, the empty set, the universal set, and the power set.
- Finite and infinite sets.
- The set operations — union, intersection, difference, complement — and the laws that govern them.
- The Cartesian product of two sets.
- How to count the size of a union of sets (inclusion–exclusion).

2.1 The Concept of a Set

Intuition

A *set* is simply a well-defined collection of objects — the mathematical version of "a bag of things". Once you fix what is in the bag, every possible object is either in it or not; there is no ambiguity. This is the same black-and-white spirit as Chapter 1, and indeed set membership and logic are two sides of the same coin, as you will see throughout this chapter.

Definition

A **set** is a well-defined collection of distinct objects, called its **elements** or **members**. We write $x \in A$ if x is an element of A, and $x \notin A$ otherwise. Sets are usually named with uppercase letters, elements with lowercase.

A set can be described in two ways:

- **Roster method** — list the elements: $B = \{\mathrm{VT}, \mathrm{RI}, \mathrm{MA}, \mathrm{CT}, \mathrm{NH}, \mathrm{ME}\}$. Order does not matter, and repeats are not counted twice: $\{x, x, y, x, y, z\} = \{x, y, z\}$.
- **Set-builder notation** — $\{x \mid P(x)\}$, read "the set of all x such that x has the property P"; the vertical bar means "such that".

Worked Example

Let B be the set of all months with exactly 30 days. Then

$B = \{x \mid x \text{ is a month with exactly 30 days}\} = \{\text{September, April, June, November}\}.$

2.1.1 Subsets, Equal Sets, Power Sets

Definition

If every element of A is also an element of B, A is a **subset** of B, denoted $A \subseteq B$. Symbolically, $(A \subseteq B) \leftrightarrow (\forall x)(x \in A \rightarrow x \in B)$. If $A \subseteq B$, we also say B **contains** A, and write $B \supseteq A$. If A is not a subset of B we write $A \nsubseteq B$; thus $(A \nsubseteq B) \leftrightarrow (\exists x)(x \in A \wedge x \notin B)$.
Two sets A and B are **equal**, $A = B$, if they have exactly the same elements: $(A = B) \leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$. If $A \subseteq B$ and $A \neq B$, then A is a **proper subset** of B, denoted $A \subset B$.

Intuition

To show $X \subseteq Y$: pick an *arbitrary* element $x \in X$, and show, using logic and known facts, that $x \in Y$ too. To show $X \nsubseteq Y$: it suffices to exhibit one element of X that is not in Y. To show $X = Y$: show $X \subseteq Y$ *and* $Y \subseteq X$.

Worked Example

Let A = the set of states in the United States, B = the set of New England states, C = the set of Canadian provinces. Then $B \subseteq A$, but $B \nsubseteq C$ and $A \nsubseteq C$.

Definition

The set with no elements is the **empty** (or **null**) **set**, denoted $\varnothing$ or $\{\}$. It is always true that $\varnothing \subseteq A$ for every set A, and there is exactly one empty set.
A special set $U\ (\neq \varnothing)$, chosen so that every set under discussion is a subset of it, is the **universal set**: $A \subseteq U$ for every set A we consider.
The **power set** of A, denoted $\mathcal{P}(A)$, is the set of *all* subsets of A (including $\varnothing$ and A itself).

Warning

$\{\varnothing\} \neq \varnothing$. The set $\{\varnothing\}$ *contains* one element (namely $\varnothing$), whereas $\varnothing$ contains no elements at all. Do not confuse "the empty box" with "a box containing an empty box".

Worked Example — Power Set

Find $\mathcal{P}(A)$ for $A = \{a, b\}$.

Solution. List every subset of A: the empty set, the two singletons, and A itself.

$$\mathcal{P}(A) = \Big\{\varnothing,\ \{a\},\ \{b\},\ \{a,b\}\Big\}.$$

Notice $|\mathcal{P}(A)| = 4 = 2^{|A|}$; a set with n elements has 2^n subsets (Section 2.3).

Intuition

Sets are often pictured with a **Venn diagram**: a rectangle represents the universal set U, and circles inside it represent sets. $A \subseteq B$ is drawn as circle A entirely inside circle B; two sets with elements in common are drawn as overlapping circles.

2.1.2 Finite and Infinite Sets

Definition

A set A is **finite** if it has exactly n elements for some non-negative integer n (this n is the **cardinality** of A, written $|A|$); otherwise A is **infinite**.

Worked Example

$\{1, 3, 5, \ldots, 99\}$ is finite, with $|A| = 50$. $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}$ are infinite sets.

Module V Extension — Not Examined

Hilbert's Hotel. Infinite sets behave very strangely compared to finite ones. Imagine a hotel with infinitely many rooms, numbered $1, 2, 3, \ldots$, and every room is occupied. A new guest arrives — can the hotel accommodate them? Yes! Simply ask every current guest to move from room n to room $n + 1$; room 1 is now free. Even more surprisingly, the hotel can make room for *infinitely many* new guests at once (move the guest in room n to room $2n$, freeing all the odd-numbered rooms). This paradox, due to David Hilbert, illustrates why infinite sets need their own careful theory (countable vs. uncountable infinity) — a topic beyond this course. This material (Koshy, Section 2.1, "Hilbert Hotel" onward) is marked optional in the syllabus and is not externally examined.

Exam Tip

Section 2.1 is examinable through Example 2.7 (power sets); Example 2.6 and everything from the Hilbert Hotel paradox onward is optional/not examined. Expect direct questions on subset notation, power sets, and Venn diagrams.

2.2 Operations with Sets

Intuition

Just as $\wedge$ and $\vee$ combine two propositions into a new one, we can combine two sets into a new set. In fact, the parallel is exact: intersection behaves like "and", union behaves like "or".

Definition

Let A, B be sets and U the universal set.

- **Union**: $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- **Intersection**: $A \cap B = \{x \mid x \in A \wedge x \in B\}$. If $A \cap B = \varnothing$, A and B are **disjoint**.
- **Difference** (relative complement of B in A): $A - B = \{x \in A \mid x \notin B\}$.
- **Complement**: $A' = U - A = \{x \in U \mid x \notin A\}$.
- **Symmetric difference**: $A \oplus B = (A \cup B) - (A \cap B)$.

Worked Example

Let $A = \{\text{Nov}, \text{Dec}, \text{Jan}, \text{Feb}\}$, $B = \{\text{Feb}, \text{Mar}, \text{Apr}, \text{May}\}$, $C = \{\text{Sept}, \text{Oct}, \text{Nov}, \text{Dec}\}$. Then $A \cap B = \{\text{Feb}\}$ and $B \cap C = \varnothing = C \cap B$ — so B and C are disjoint. (Two sets are disjoint iff their intersection is null.)

Worked Example — Combining Operations

Let $A = \{a, \ldots, z, 0, \ldots, 9\}$ and $B = \{0, \ldots, 9\}$. Then $A - B = \{a, \ldots, z\}$ (all the letters) and $B - A = \varnothing$ (every digit is already in A). Note that $A - B \neq B - A$ in general — order matters.

Worked Example — Set Identities via Venn Diagrams

Let $A = \{a, b, c, d, g\}$, $B = \{b, c, d, e, f\}$, $C = \{b, c, e, g, h\}$. Find $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.

Solution.

$$\begin{aligned} B \cap C &= \{b, c, e\} \\ A \cup (B \cap C) &= \{a, b, c, d, e, g\} \end{aligned}$$

$$\begin{aligned} A \cup B &= \{a, b, c, d, e, f, g\} \\ A \cup C &= \{a, b, c, d, e, g, h\} \\ (A \cup B) \cap (A \cup C) &= \{a, b, c, d, e, g\} \ = \ A \cup (B \cap C) \end{aligned}$$

This is not a coincidence — it is the distributive law (Law 14 below).

2.2.1 The Laws of Sets

Definition

Let A, B, C be any sets and U the universal set. The following identities always hold (Koshy, Table 2.2):

Idempotent laws	$A \cup A = A$	$A \cap A = A$
Identity laws	$A \cup \varnothing = A$	$A \cap U = A$
Inverse laws	$A \cup A' = U$	$A \cap A' = \varnothing$
Domination laws	$A \cup U = U$	$A \cap \varnothing = \varnothing$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Double complementation		$(A')' = A$
Associative laws	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
Distributive laws		$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
		$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
De Morgan's laws	$(A \cup B)' = A' \cap B'$	$(A \cap B)' = A' \cup B'$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

(The following have no standard names.)

If $A \subseteq B$: $A \cap B = A$ $A \cup B = B$ $B' \subseteq A'$

Difference identity $A - B = A \cap B'$

Symmetric difference $A \oplus B = (A \cup B) - (A \cap B)$

Key Takeaway

The laws of sets and the laws of logic (Section 1.2) are structurally identical: $\cup \leftrightarrow \vee$, $\cap \leftrightarrow \wedge$, $(\,)' \leftrightarrow \sim$, $\varnothing \leftrightarrow f$, $U \leftrightarrow t$. If you know one table, you effectively know the other.

Worked Example — Proving a Set Identity

Prove that $A' \cap B' \subseteq (A \cup B)'$.
Solution. Let x be any element of $A' \cap B'$. Then $x \in A'$ and $x \in B'$, so $x \notin A$ and $x \notin B$. By De Morgan's law (for logic), $x \notin (A \cup B)$, i.e., $x \in (A \cup B)'$. Since x was arbitrary, $A' \cap B' \subseteq (A \cup B)'$. (Combined with the reverse inclusion, this gives the set-theoretic De Morgan's law $(A \cup B)' = A' \cap B'$.)

Worked Example — Simplifying a Set Expression

Simplify $(A \cap B') \cup (A' \cap B) \cup (A' \cap B')$. [Oct 2024]
Solution. This expression describes every region of the Venn diagram *except* $A \cap B$

(it is the complement of $A \cap B$):

$$\begin{aligned}
(A \cap B') \cup (A' \cap B) \cup (A' \cap B') &= (A \cap B') \cup \left[A' \cap (B \cup B')\right] && \text{distributive} \\
&= (A \cap B') \cup (A' \cap U) && \text{inverse law} \\
&= (A \cap B') \cup A' && \text{identity law} \\
&= (A \cup A') \cap (B' \cup A') && \text{distributive} \\
&= U \cap (A' \cup B') && \text{inverse law} \\
&= A' \cup B' \;=\; (A \cap B)' && \text{identity law, De Morgan}
\end{aligned}$$

2.2.2 The Cartesian Product

Definition

The **Cartesian product** of sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

$A \times A$ is denoted A^2. Named after René Descartes.

Worked Example

Let $A = \{a, b\}$, $B = \{x, y, z\}$.

$$\begin{aligned}
A \times B &= \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\} \\
B \times A &= \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\} \\
A^2 = A \times A &= \{(a, a), (a, b), (b, a), (b, b)\}
\end{aligned}$$

Notice $A \times B \neq B \times A$ — order matters, exactly as with ordered pairs.

Warning

$|A \times B| = |A| \cdot |B|$, *not* $|A| + |B|$. This is the same "multiply for sequential choices" idea you will meet formally as the Fundamental Counting Principle in Chapter 4.

Exam Tip

Section 2.2 is examinable through Example 2.21, plus the Cartesian product; the material on fuzzy sets and fuzzy operations is optional/not examined. Simplifying a set expression using the 24 laws (Section B/C style) and listing elements of a Cartesian product (Section A style, [Oct 2024]) are both common questions.

2.3 The Cardinality of a Set

Intuition

If two sets overlap, simply adding their sizes double-counts the overlap. Fixing this double-counting is the single idea behind everything in this section, and it generalises beautifully to three or more sets.

Definition

For finite sets A, B:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

More generally, for three finite sets A, B, C (the **principle of inclusion–exclusion**):

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Intuition

Why the alternating $+, -, +$ pattern? Adding $|A| + |B| + |C|$ counts every element in exactly one set once, every element in exactly two sets *twice*, and every element in all three sets *three times*. Subtracting the pairwise intersections removes the double-counting, but now removes the triple-overlap element three times total (once per pair) against its original three counts — net zero — so it must be added back once.

Worked Example — Two Sets

Let $|A| = 3, |B| = 5, |A \cap B| = 2$. Find $|A \cup B|$. [Oct 2024]
Solution. $|A \cup B| = |A| + |B| - |A \cap B| = 3 + 5 - 2 = 6$.

Worked Example — Three Sets

$|A| = 10, |B| = 15, |C| = 20, |A \cap B| = 5, |A \cap C| = 4, |B \cap C| = 3, |A \cap B \cap C| = 2$.
Find $|A \cup B \cup C|$. [Oct 2024]
Solution.

$$|A \cup B \cup C| = 10 + 15 + 20 - 5 - 4 - 3 + 2 = 35.$$

Worked Example — Counting with a Venn Diagram

Find the number of positive integers ≤ 3000 that are *not* divisible by 7 or 8. [Oct 2024]
Solution. Let A = multiples of 7 up to 3000, B = multiples of 8 up to 3000. Then $|A| = \lfloor 3000/7 \rfloor = 428$, $|B| = \lfloor 3000/8 \rfloor = 375$. A number divisible by both 7 and 8 is divisible by $\mathrm{lcm}(7, 8) = 56$: $|A \cap B| = \lfloor 3000/56 \rfloor = 53$.

$$|A \cup B| = 428 + 375 - 53 = 750.$$

So the count *not* divisible by 7 or 8 is $3000 - 750 = 2250$.

Module V Extension — Not Examined

Counting subsets (Koshy, Theorem 2.2). If a set S has n elements, it has exactly 2^n subsets. *Idea of proof:* let s_n be the number of subsets of an n-element set, and remove one element x to get an $(n-1)$-element set S^* with s_{n-1} subsets. Every subset of S^* is a subset of S; adding x to each gives another s_{n-1} subsets of S that contain x. Since every subset of S either contains x or does not, $s_n = 2s_{n-1}$, which unwinds to $s_n = 2^n$. This theorem and its algorithmic proof are marked optional in the syllabus and not externally examined — but the *formula* $|\mathcal{P}(A)| = 2^{|A|}$ itself is worth remembering (it also follows directly from the Fundamental Counting Principle in Chapter 4: each of the n elements is independently either "in" or "out".)

Exam Tip

Two-set cardinality and three-set inclusion–exclusion are both squarely examinable and appeared multiple times on the October 2024 paper. Theorem 2.2 (subset-counting) is optional/not examined — know the formula 2^n, but you will not be asked to reproduce its proof.

Module V Extension — Relations and Digraphs

Module V Extension — Not Examined

Relations. Now that we have the Cartesian product $A \times B$, we can formalise the idea of a "relationship" between elements of two sets.

Definition. A **relation** R from a set A to a set B is simply a subset of $A \times B$: $R \subseteq A \times B$. If $(a, b) \in R$ we write $a\,R\,b$ ("a is related to b"). A relation *from* A *to* A is called a relation **on** A.

Example. Let $A = \{1, 2, 3\}$ and define R on A by $a\,R\,b$ iff $a \leq b$. Then

$$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\} \subseteq A \times A.$$

Digraphs. A relation R on a finite set A can be drawn as a **directed graph** (**digraph**): each element of A is a vertex (dot), and we draw an arrow from a to b whenever $a\,R\,b$. For the example above, vertices $1, 2, 3$ each get a self-loop (since $a \leq a$), plus arrows $1 \to 2$, $1 \to 3$, $2 \to 3$.

Properties of a relation on A. R is:

- **reflexive** if $a\,R\,a$ for every $a \in A$ (every vertex has a self-loop);
- **symmetric** if $a\,R\,b \Rightarrow b\,R\,a$ (every arrow has a matching arrow the other way);
- **transitive** if $a\,R\,b$ and $b\,R\,c$ together imply $a\,R\,c$ ("shortcut" arrows are already present).

A relation that is reflexive, symmetric, and transitive is an **equivalence relation** — it partitions A into disjoint groups of "equivalent" elements. (The relation $\leq$ above is reflexive and transitive, but *not* symmetric.)

This topic is part of the syllabus's open-ended Module V and is **not** externally examined, but it directly extends the Cartesian product you just learned, and it is the natural next step if you continue into graph theory or discrete structures.

Chapter Summary

- A **set** is a well-defined collection; described by roster or set-builder notation. $A \subseteq B$, $A \subset B$, $A = B$, $\varnothing$, U, $\mathcal{P}(A)$.
- **Operations**: $A \cup B$, $A \cap B$, $A - B$, $A' = U - A$, $A \oplus B$; governed by the 24 laws of Table 2.2, which mirror the laws of logic exactly.
- **Cartesian product**: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$, $|A \times B| = |A||B|$.
- **Cardinality**: $|A \cup B| = |A| + |B| - |A \cap B|$; inclusion–exclusion for three sets; $|\mathcal{P}(A)| = 2^{|A|}$.
- *Module V extension:* relations $R \subseteq A \times B$, digraphs, reflexive/symmetric/transitive properties.

Practice Problems

1. Write $\{x \mid x$ is a letter in the word "MALAYALAM"$\}$ using the roster method.
2. Let $A = \{a, b, x, y, z\}$, $B = \{c, d, e, x, y, z\}$, $U = \{a, b, c, d, e, w, x, y, z\}$. Find $(A \cup B)'$ and $A' \cap B'$. What do you notice?
3. Find $\mathcal{P}(A)$ for $A = \{1, 2, 3\}$. How many elements does it have?
4. Let $A = \{1, 2\}$, $B = \{a, b, c\}$. List $A \times B$ and $B \times A$. Are they equal as sets?
5. Simplify $(A \cap B') \cup (A' \cap B') \cup (A \cap B)$ using the laws of sets.
6. $|A| = 3, |B| = 5, |A \cap B| = 2$. Find $|A \cup B|$ and $|A \times B|$.
7. In a class of 100 students, 30 like Vanilla, 43 like Chocolate, and so on (construct your own three-set inclusion–exclusion problem with given pairwise and triple overlaps, then find how many like none of the three).
8. Prove, using an arbitrary-element argument (not a Venn diagram), that $A \cap B \subseteq A$.
9. Find the number of positive integers ≤ 2000 that are divisible by neither 5 nor 9.
10. Give an example of a relation on $A = \{1, 2, 3, 4\}$ that is symmetric but not reflexive, and draw its digraph.
11. Give an example of a relation on $A = \{1, 2, 3\}$ that is reflexive and transitive but not symmetric, and identify which axiom of equivalence relations it fails.
12. Using the laws of sets, show that $A - (B \cup C) = (A - B) \cap (A - C)$.

Chapter 3

Functions and Matrices

What you will learn in this chapter.

- What a function is, and the vocabulary around it: domain, codomain, image, pre-image.
- How to build new functions from old ones (piecewise definitions, sums, products).
- Special functions used throughout discrete mathematics: polynomial, exponential, logarithmic, floor/ceiling, characteristic, mod, and div.
- Matrices: notation, equality, and the arithmetic of addition, scalar multiplication, and multiplication.

3.1 The Concept of a Function

Intuition

A function is a rule that assigns, to each input, *exactly one* output — like a vending machine: press button B4, and you always get the same snack, and something always comes out. You may recognise this as exactly the "pairing" idea behind $A \times B$ from Chapter 2: a function is a very disciplined relation.

Definition

A **function** f from a set X to a set Y, written $f : X \to Y$, is an assignment of *exactly one* element of Y to each element of X. We call X the **domain**, written $\mathrm{Dom}(f)$, and Y the **codomain**, written $\mathrm{codom}(f)$. If f assigns $y \in Y$ to $x \in X$, we write $y = f(x)$; y is the **value** (or **image**) of f at x, and x is a **pre-image** of y. Read $f(x)$ as "f of x".

Warning

The definition does *not* forbid two different elements of X from being sent to the same element of Y, and it does *not* require every element of Y to be used. The only rule is: *every* $x \in X$ gets *exactly one* y. An assignment that leaves some x unassigned, or assigns some x to two different values, is **not** a function.

Worked Example — Is It a Function?

Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4\}$.

- The assignment $a \mapsto 1, b \mapsto 3, c \mapsto 2, d \mapsto 4$ **is** a function: every element of X has exactly one image. Here $\mathrm{Dom}(f) = X$ and $\mathrm{codom}(f) = Y$.
- An assignment where some element of X has *no* arrow to Y is **not** a function.
- An assignment where some element of X has *two* arrows to Y is **not** a function — it is ambiguous.

3.1.1 Piecewise Functions, and New Functions from Old

Definition

A function may be defined by different formulas on different parts of its domain — a **piecewise** definition.

Worked Example

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -1 & \text{if } -2 \leq x < 0 \\ 3x + 4 & \text{otherwise} \end{cases}$$

is a single function $g : \mathbb{R} \to \mathbb{R}$, built from three formulas covering all of $\mathbb{R}$ between them.

Definition

Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. The **sum** and **product** of f and g are

$$(f + g)(x) = f(x) + g(x), \qquad (fg)(x) = f(x) \cdot g(x),$$

defined wherever *both* f and g are defined: $\mathrm{Dom}(f + g) = \mathrm{Dom}(fg) = \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$.
Two functions $f : A \to B$ and $g : C \to D$ are **equal** if $A = C$, $B = D$, and $f(x) = g(x)$ for every $x \in A$.

Worked Example — Sum and Product of Functions

Let $f(x) = x^2$ (so $\mathrm{Dom}(f) = (-\infty, \infty)$) and $g(x) = \sqrt{x - 1}$ (so $\mathrm{Dom}(g) = [1, \infty)$). Then

$$(f + g)(x) = x^2 + \sqrt{x - 1}, \qquad (fg)(x) = x^2\sqrt{x - 1},$$

and since $\mathrm{Dom}(f) \cap \mathrm{Dom}(g) = [1, \infty)$, both new functions have domain $[1, \infty)$.

Exam Tip

Section 3.1 is examinable through Example 3.2, plus piecewise/sum/product functions; Example 3.7 (the ASCII piecewise example) is optional. Expect definition-recall questions and short computations of $(f + g)(x)$ or $(fg)(x)$ given two explicit functions.

3.2 Special Functions

Intuition

Discrete mathematics leans on a handful of “workhorse” functions again and again — much as calculus leans on polynomials and trig functions. This section collects the ones you will meet throughout the rest of the course.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ (with $a_n \neq 0$) is a **polynomial function** of degree n. When $n = 1$, f is **linear**; when $n = 2$, f is **quadratic**.
Let $a \in \mathbb{R}^+$, $a \neq 1$. The function $f(x) = a^x$ is an **exponential function** with base a. If $y = a^x$, then x is the **logarithm** of y to base a, written $x = \log_a y$; thus $(\log_a y = x) \leftrightarrow (y = a^x)$.

3.2.1 Floor and Ceiling Functions

Definition

For a real number x, the **floor** of x, written $\lfloor x \rfloor$, is the largest integer $\leq x$. The **ceiling** of x, written $\lceil x \rceil$, is the smallest integer $\geq x$.

Worked Example

$\lfloor 3.7 \rfloor = 3$, $\lfloor -3.7 \rfloor = -4$, $\lceil 3.2 \rceil = 4$, $\lceil -3.2 \rceil = -3$, $\lfloor 5 \rfloor = \lceil 5 \rceil = 5$.

Intuition

The post-office function. In 2003, first-class postage was 37¢ for up to 1 ounce, plus 23¢ for each additional ounce or fraction thereof, up to 11 ounces: $p(x) = 0.37 + 0.23\lceil x - 1 \rceil$ for $0 < x \leq 11$. A letter weighing 7.8 ounces costs $p(7.8) = 0.37 + 0.23\lceil 6.8 \rceil = 0.37 + 0.23(7) = \$1.98$ — the ceiling function captures “rounds up to the next whole ounce” exactly.

Key Properties (Theorem 3.1)

For any real x and integer n:

$$\lfloor n \rfloor = n = \lceil n \rceil, \qquad \lfloor x + n \rfloor = \lfloor x \rfloor + n, \qquad \lceil x + n \rceil = \lceil x \rceil + n,$$

and if $n \notin \mathbb{Z}$, $\lceil x \rceil = \lfloor x \rfloor + 1$. (Proof omitted — not examined, per the syllabus.)

3.2.2 Characteristic Function

Definition

Let U be a universal set and $S \subseteq U$. The **characteristic function** of S is $f_S : U \to \{0, 1\}$,

$$f_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases}$$

Worked Example

$U = \{a, b, c, d, e, f\}$, $A = \{a, c, d, e\}$. Then $f_A(a) = f_A(c) = f_A(d) = f_A(e) = 1$ and $f_A(b) = f_A(f) = 0$. Listing the bits right-to-left, A corresponds to the 6-bit word 011101 — so a subset of an n-element universe can always be represented as an n-bit word.

Theorem 3.3 — Properties of the Characteristic Function

For any sets $A, B \subseteq U$ and $x \in U$:

$$f_{A\cap B}(x) = f_A(x) \cdot f_B(x), \qquad f_{A\cup B}(x) = f_A(x) + f_B(x) - f_{A\cap B}(x),$$

$$f_{A'}(x) = 1 - f_A(x), \qquad f_{A\oplus B}(x) = f_A(x) + f_B(x) - 2f_{A\cap B}(x).$$

Notice the strong resemblance to the cardinality formula $|A\cup B| = |A|+|B|-|A\cap B|$ from Section 2.3 — both come from the same "avoid double counting" idea, one at the level of individual elements (0/1) and one at the level of set sizes.

3.2.3 Mod and Div Functions

Definition

For an integer x and positive integer y: the **mod function** $f(x, y) = x \bmod y$ is the *remainder* when x is divided by y; the **div function** $g(x, y) = x \operatorname{div} y$ is the *quotient*.

Worked Example

$23 \bmod 5 = 3$ (since $23 = 4 \cdot 5 + 3$), $18 \bmod 6 = 0$, $23 \operatorname{div} 5 = 4$, $5 \operatorname{div} 6 = 0$.

Worked Example — Days of the Week

Today is Thursday. What day will it be in 100 days?
Solution. The mod function finds the day of the week n days from today by removing complete weeks: $100 \bmod 7 = 2$. Two days after Thursday is **Saturday**.

Exam Tip

Section 3.2 is examinable through Example 3.13 (special functions, including characteristic/mod/div), plus the floor/ceiling material; the proofs of Theorems 3.1 and 3.2, Theorem 3.3's proof, the *Card Dealing* application, and the *Two Queens Puzzle* are all optional/not examined. Know the definitions and be able to compute values (e.g., a mod/div computation, or a characteristic-function table) — do not worry about proving the identities.

3.3 Matrices

Intuition

A matrix is just a rectangular table of numbers — the same kind of table you already use for a timetable, a mark-list, or a spreadsheet. What makes matrices powerful is that we can define *arithmetic* on these tables (add them, scale them, multiply them), and that arithmetic turns out to encode enormous amounts of mathematics, from solving systems of equations (see the Module V extension below) to computer graphics.

Definition

A **matrix** is a rectangular arrangement of numbers enclosed in brackets. A matrix with m rows and n columns is an $m \times n$ ("m by n") matrix; this is its **size**. If $m = 1$ it is a **row vector**; if $n = 1$, a **column vector**; if $m = n$, a **square matrix of order** n. Each number is an **element** (or **entry**). Matrices are named with uppercase letters.
Using **double subscript notation**, a_{ij} denotes the element in row i, column j of A; we abbreviate $A = (a_{ij})_{m\times n}$, or simply (a_{ij}) when the size is clear.

Worked Example

$$A = \begin{bmatrix} 3 & -5 & 6 \\ 1 & 0 & 4 \end{bmatrix}$$

is a 2×3 matrix; $a_{12} = -5$, $a_{23} = 4$.

Definition

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are **equal** if they have the same size and $a_{ij} = b_{ij}$ for every i, j.
If every entry of a matrix is 0, it is the **zero matrix**, O. In a square matrix $A = (a_{ij})_{n\times n}$, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ form the **main diagonal**. If $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ otherwise, A is the **identity matrix** I_n.

3.3.1 Matrix Addition and Scalar Multiplication

Definition

For matrices of the *same* size $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$, and scalar $c \in \mathbb{R}$:

$$A + B = (a_{ij} + b_{ij})_{m\times n}, \qquad cA = (ca_{ij})_{m\times n}.$$

The **negative** of A is $(-1)A = -A$.

Worked Example — Matrix Addition

$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & -1 \end{bmatrix}$. [Oct 2024]

$$A + B = \begin{bmatrix} 2+1 & -3+5 & 7+0 \\ 0+2 & 1+0 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 1 & 0 \end{bmatrix}.$$

Theorem 3.12 — Properties of Matrix Arithmetic

For any $m \times n$ matrices A, B, C, the $m \times n$ zero matrix O, and scalars c, d:

$$\begin{array}{ll} A + B = B + A & A + (B + C) = (A + B) + C \\ A + O = A = O + A & A + (-A) = O = (-A) + A \\ (-1)A = -A & c(A + B) = cA + cB \\ (c + d)A = cA + dA & (cd)A = c(dA) \end{array}$$

(Proof omitted — not examined, per the syllabus. It is a routine check, entry by entry, using the corresponding properties of real numbers.)

3.3.2 Matrix Multiplication

Definition

The **product** AB of $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$ is the matrix $C = (c_{ij})_{m\times p}$, where c_{ij} is obtained by multiplying corresponding entries in row i of A and column j of B, and adding:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Warning

The product AB is defined **only if** the number of columns of A equals the number of rows of B; the result has size (rows of A) $\times$ (columns of B). In particular, *even when both AB and BA are defined, they are generally not equal* — matrix multiplication is not commutative.

Worked Example — Matrix Multiplication

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}$ (2×3) and $B = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$ (3×2). Find AB and BA, if defined. [Oct 2024]

Solution. A has 3 columns, B has 3 rows, so AB is defined and is 2×2:

$$AB = \begin{bmatrix} 1(3) + (-2)(0) + 3(-1) & 1(-2) + (-2)(1) + 3(0) \\ 0(3) + 4(0) + (-1)(-1) & 0(-2) + 4(1) + (-1)(0) \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix}.$$

B has 2 columns, A has 2 rows, so BA is also defined, and is 3×3:

$$BA = \begin{bmatrix} 3(1) + (-2)(0) & 3(-2) + (-2)(4) & 3(3) + (-2)(-1) \\ 0(1) + 1(0) & 0(-2) + 1(4) & 0(3) + 1(-1) \\ -1(1) + 0(0) & -1(-2) + 0(4) & -1(3) + 0(-1) \end{bmatrix} = \begin{bmatrix} 3 & -14 & 11 \\ 0 & 4 & -1 \\ -1 & 2 & -3 \end{bmatrix}.$$

Notice AB (2×2) and BA (3×3) are not even the same *size*, let alone equal — a vivid illustration of the warning above.

Exam Tip

Matrix addition, scalar multiplication, and multiplication (including checking whether AB/BA are defined, and computing them) are core exam material, and appeared repeatedly on the October 2024 paper — including this exact example. Theorem 3.12's proof and the algorithm-analysis of matrix product are optional/not examined.

Module V Extension — Basic Calculus Concepts

Module V Extension — Not Examined

Functions are the common language between this course and **MAT1CJ101 — Differential Calculus**, which you are likely taking concurrently. Four ideas from calculus are worth previewing here, purely as orientation (they are developed properly, with full rigour, in that course):

- **Limit.** $\lim_{x \to a} f(x) = L$ means $f(x)$ gets arbitrarily close to L as x gets arbitrarily close to a.
- **Continuity.** f is continuous at a if $\lim_{x \to a} f(x) = f(a)$ — informally, the graph of f has no "jump" or "hole" at $x = a$. The piecewise function $g(x)$ from Section 3.1 is *not* continuous at $x = 0$ if the three pieces do not meet up.
- **Differentiation.** The derivative $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ measures the instantaneous rate of change (slope) of f at a.
- **Integration.** $\int_a^b f(x)\,dx$ measures the (signed) area between the graph of f and the x-axis, from $x = a$ to $x = b$.

This is Module V, open-ended material and **not** externally examined in

MAT1MN104.

Module V Extension — Root-Finding: Bisection and Regula-Falsi Methods

Module V Extension — Not Examined

Given a continuous function f, a **root** is a value r with $f(r) = 0$. Many equations cannot be solved exactly by algebra, so we approximate roots numerically.

Bisection method. If f is continuous and $f(a)$, $f(b)$ have opposite signs, a root lies somewhere in $[a, b]$ (Intermediate Value Theorem). Repeatedly bisect: let $c = \dfrac{a+b}{2}$; if $f(c) = 0$, done; if $f(a)$ and $f(c)$ have opposite signs, the root is in $[a, c]$ (replace b by c); otherwise it is in $[c, b]$ (replace a by c). Repeat until the interval is as small as desired. This always converges (for continuous f), but can be slow.

Regula-Falsi (false position) method. Same setup as bisection, but instead of the midpoint, use the point where the *straight line* through $(a, f(a))$ and $(b, f(b))$ crosses the x-axis:

$$c = \frac{a\, f(b) - b\, f(a)}{f(b) - f(a)}.$$

This usually converges faster than plain bisection, since it uses the function's actual values (not just their signs) to make a smarter guess.
This is Module V, open-ended material and **not** externally examined in MAT1MN104.

Module V Extension — The Gauss–Jordan Method

Module V Extension — Not Examined

Matrices are the natural language for solving a system of linear equations. Write the system as an **augmented matrix** $[A \mid \mathbf{b}]$, e.g.

$$\begin{cases} x - 2y + 3z = 1 \\ 4y - z = 2 \end{cases} \qquad \longleftrightarrow \qquad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 4 & -1 & 2 \end{array}\right].$$

The **Gauss–Jordan method** applies **elementary row operations** — (1) swap two rows, (2) multiply a row by a nonzero scalar, (3) add a multiple of one row to another — to reduce $[A \mid \mathbf{b}]$ to **reduced row-echelon form**: leading 1's ("pivots") in a staircase pattern, with 0's everywhere else in each pivot column. Once in this form, the solution can be read off directly.

Worked example. Solve $\begin{cases} x+y=3 \\ 2x-y=0 \end{cases}$ by Gauss–Jordan.

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -6 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Reading off: $x = 1$, $y = 2$. (Check: $1 + 2 = 3$ ✓, $2(1) - 2 = 0$ ✓.)
This method is a direct extension of the matrix operations you just learned, and is Module V, open-ended material — **not** externally examined in MAT1MN104.

Chapter Summary

- A **function** $f : X \to Y$ assigns exactly one $y = f(x) \in \text{codom}(f)$ to each $x \in \text{Dom}(f)$.
- Functions can be **piecewise**; two functions combine via $(f + g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$, with domain $\text{Dom}(f) \cap \text{Dom}(g)$.
- **Special functions**: polynomial, exponential/logarithmic, floor $\lfloor x \rfloor$/ceiling $\lceil x \rceil$, characteristic function f_S, mod and div.
- A **matrix** $A = (a_{ij})_{m\times n}$; addition and scalar multiplication are entrywise; the product AB (defined only if columns of A = rows of B) uses row-times-column dot products, and is *not* commutative.
- *Module V extensions:* calculus preview (limits, continuity, derivatives, integrals), bisection/regula-falsi root-finding, Gauss–Jordan elimination.

Practice Problems

1. Which of the following are functions from $\{1, 2, 3\}$ to $\{a, b\}$? (i) $1 \to a, 2 \to b, 3 \to a$. (ii) $1 \to a, 2 \to a$ (3 unassigned). (iii) $1 \to a, 1 \to b, 2 \to a, 3 \to b$.
2. Let $f(x) = x^2 - 1$, $g(x) = \sqrt{x}$ with $\text{Dom}(g) = [0, \infty)$. Find $(f + g)(x)$ and $(fg)(x)$ and state their domain.
3. Define the absolute value function piecewise and sketch its graph.
4. Compute $\lfloor -2.3 \rfloor$, $\lceil -2.3 \rceil$, $\lfloor 7 \rfloor$, $17 \bmod 5$, $17 \operatorname{div} 5$.
5. $U = \{1, \dots, 8\}$, $A = \{1, 3, 5, 7\}$, $B = \{2, 3, 5, 7\}$. Write out f_A, f_B, and verify $f_{A\cap B}(x) = f_A(x)f_B(x)$ for $x = 3$ and $x = 4$.
6. If today is Wednesday, what day will it be in 250 days? Show your mod computation.
7. Let $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix}$. Find $A + B$, $3A$, AB, and BA. Is $AB = BA$ here?
8. Let A be 2×3 and B be 3×4. Is AB defined? Is BA defined? Give the size of whichever product(s) exist.
9. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Verify $A + (-A) = O$ directly.

10. (Module V) Use one step of the bisection method to narrow down a root of $f(x) = x^3 - x - 2$ known to lie in $[1, 2]$.
11. (Module V) Solve $\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$ using the Gauss–Jordan method, showing each row operation.
12. Prove, using the definition of matrix equality, that if $A + B = A$ for all $m \times n$ matrices A, then B must be the $m \times n$ zero matrix.

Chapter 4

Combinatorics and Discrete Probability

What you will learn in this chapter.

- The fundamental counting principles — when to add, and when to multiply.
- Permutations: counting ordered arrangements, including arrangements in a circle.
- Combinations: counting unordered selections.
- The basics of discrete probability, built directly on top of counting.

4.1 The Fundamental Counting Principles

Intuition

Counting is deceptively hard to do by listing — try listing all $7! = 5040$ ways to arrange 7 books on a shelf! Instead, we count *indirectly*, by breaking a big counting problem into smaller tasks and combining their counts with two simple rules: add when you choose *one task or another*, multiply when you do *one task and then another*.

Addition Principle

If a task can be done in m ways and a second, *mutually exclusive* task (no overlap) can be done in n ways, then either task can be done in $m+n$ ways. More generally, if pairwise mutually exclusive tasks $T_1, \ldots, T_k$ can be done in $m_1, \ldots, m_k$ ways respectively, then "T_1, or T_2, ..., or T_k" can be done in $m_1 + m_2 + \cdots + m_k$ ways.

Worked Example

A student needs one more course, eligible to take 15 English, 10 French, or 6 German courses (no course counted in more than one category). By the addition principle, there are $15 + 10 + 6 = 31$ choices.

Theorem 6.2 — Inclusion–Exclusion Principle (for tasks)

If task A can be done in m ways, task B in n ways, and *both* can be accomplished simultaneously in k ways, then "task A or task B" can be done in $m + n - k$ ways.

Worked Example

In how many ways can you deal a king or a black card from a standard deck? A king can be chosen in 4 ways, a black card in 26 ways; both at once (a black king) in 2 ways. So a king or black card can be dealt in $4 + 26 - 2 = 28$ ways.

Theorem 6.3 — Multiplication Principle

Suppose a task T consists of subtask T_1 *followed by* subtask T_2. If T_1 can be done in m_1 ways, and for *each* of those, T_2 can be done in m_2 ways, then T can be done in $m_1 m_2$ ways. (This extends to any number of sequential subtasks: multiply all the counts.)

Intuition

This is exactly why $|A \times B| = |A| \cdot |B|$ (Section 2.2): choosing an ordered pair (a, b) is "choose a, *then* choose b" — two sequential subtasks.

Worked Example — Multiplication Principle

A password consists of 2 letters followed by 3 digits. How many passwords are possible (repetition allowed)?
Solution. Five sequential choices: letter (26 ways), letter (26), digit (10), digit (10), digit (10). By the multiplication principle,

$$26 \times 26 \times 10 \times 10 \times 10 = 676{,}000.$$

Exam Tip

Section 6.1 is examinable except Example 6.7. Distinguishing "and" (multiply) from "or, mutually exclusive" (add) from "or, overlapping" (inclusion–exclusion, subtract the overlap) is the single most important skill here — and it resurfaces throughout the rest of the chapter.

4.2 Permutations

Intuition

A **permutation** is simply an *ordered* arrangement. The words *abc* and *acb* use the same three letters but are different permutations, because order matters. (Contrast this with combinations, Section 4.3, where order will *not* matter.)

Definition

An ordered arrangement of r elements chosen from a set of n distinct elements is an r-**permutation**; the number of such arrangements is denoted $P(n, r)$.

Theorem 6.4

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Intuition

Why? There are n elements, so the first position can be filled in n ways. Once used, $n-1$ elements remain for the second position, then $n-2$ for the third, and so on, down to $n-r+1$ ways for the rth position. By the multiplication principle,

$$\begin{aligned} P(n, r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)\cdots(n-r+1)\cdot(n-r)(n-r-1)\cdots 2\cdot 1}{(n-r)(n-r-1)\cdots 2\cdot 1} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

Warning

For large n, compute $P(n, r)$ as $n(n-1)\cdots(n-r+1)$ directly, *not* by first computing $n!$ and $(n-r)!$ separately and dividing — both factorials may be far too large for a calculator, even though their ratio is small. For example, $P(25, 5) = \frac{25!}{20!} = 25\cdot 24\cdot 23\cdot 22\cdot 21 = 6{,}375{,}600$.

Worked Example

Find the number of words that can be formed by scrambling the letters of the word SCRAMBLE. [Oct 2024]

Solution. SCRAMBLE has 8 *distinct* letters, so we want the number of ways to arrange all 8: $P(8, 8) = 8! = 40{,}320$.

4.2.1 Cyclic Permutations

Intuition

Arranging beads on a necklace, or people around a round table, behaves differently from a straight line: rotating the whole arrangement gives the “same” arrangement back, since there is no fixed starting point.

Theorem 6.6

The number of cyclic permutations of n distinct items (arrangements around a circle, where rotations are considered identical) is $(n-1)!$.

Intuition

Why? Fix the position of the first item around the circle (this removes the "which rotation" ambiguity). The remaining $n-1$ items can now be placed in the remaining $n-1$ positions in any order: $(n-1)!$ ways, by the multiplication principle applied to a linear arrangement of what's left.

Worked Example

Five zinnias planted in a circle: $(5-1)! = 4! = 24$ distinct circular arrangements (compare with $5! = 120$ arrangements in a straight line).

Exam Tip

Section 6.2 is examinable through Example 6.13, plus cyclic permutations; the proof of Theorem 6.4 and the "Fibonacci numbers revisited" material are optional/not examined. Expect direct application of $P(n,r) = n!/(n-r)!$ and of $(n-1)!$ for circular arrangements.

4.3 Combinations

Intuition

Sometimes order genuinely does not matter: a committee {Costa, Shea, Weiss} is the same committee no matter what order you list the names in. Counting *unordered* selections is the subject of this section.

Definition

An unordered selection of r elements from a set of n distinct elements is an r-**combination**; the number of such selections is denoted $C(n,r)$.

Deriving the Formula

Every r-combination gives rise to $P(r,r) = r!$ different r-permutations (arrange its r elements in every possible order). So the total number of r-permutations is $r! \cdot C(n,r)$. But we already know $P(n,r) = n!/(n-r)!$ counts the r-permutations directly. Equating,

$$r!\,C(n,r) = \frac{n!}{(n-r)!} \quad\Longrightarrow\quad C(n,r) = \frac{n!}{r!\,(n-r)!}.$$

Warning

As with $P(n,r)$, avoid computing huge factorials directly:

$$C(n,r) = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

Special values: $C(n,0) = C(n,n) = 1$ (there is exactly one way to choose nothing, and exactly one way to choose everything).

Worked Example

Compute the number of subcommittees of three members each that can be formed from a committee of 25 members.
Solution. Order does not matter within a subcommittee, so this is a combination:

$$C(25,3) = \frac{25 \cdot 24 \cdot 23}{3!} = \frac{13{,}800}{6} = 2{,}300.$$

Worked Example — Combinations with a Condition

Find the number of groups that can be formed from a group of seven marbles if each group must contain *at least* three marbles. [Oct 2024]
Solution. We need groups of size 3, 4, 5, 6, or 7, and these are mutually exclusive cases, so (addition principle):

$$C(7,3) + C(7,4) + C(7,5) + C(7,6) + C(7,7) = 35 + 35 + 21 + 7 + 1 = 99.$$

Exam Tip

Section 6.4 is examinable through the formula and its direct applications; the proof of Theorem 6.10 (an alternate derivation), Example 6.22, Theorem 6.12, and Example 6.26 are optional/not examined. The single most common exam pitfall is using $P(n,r)$ where $C(n,r)$ was needed, or vice versa — always ask first: *does order matter here?*

4.4 Discrete Probability

Intuition

Probability answers the question "out of all the equally likely things that could happen, what fraction of them are the outcome I care about?" — and that fraction is computed with exactly the counting tools (permutations, combinations) you just built.

Definition

The **sample space** S of an experiment is the set of all its possible outcomes (assumed finite and nonempty). An **event** E is a subset of S. An outcome in E is

favorable; one not in E is **unfavorable**. If $E = \varnothing$, E is an **impossible event**; if $|E| = 1$, E is a **simple event**. The **complement** of E is $E' = S - E$.

Worked Example

Tossing three coins: by the multiplication principle, $|S| = 2 \cdot 2 \cdot 2 = 8$, so

$$S = \{\text{HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}\}.$$

Let A = "exactly two heads" = $\{\text{HHT,HTH,THH}\}$, B = "at least two heads" = $\{\text{HHT,HTH,THH,HHH}\}$, C = "four heads" = $\varnothing$ (impossible, since we only tossed three coins).

Probability of an Event

Let E be an event of a finite sample space S with *equally likely* outcomes. The **probability** of E is

$$p(E) = \frac{|E|}{|S|} = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}}.$$

Worked Example

A card is drawn at random from a standard deck. Find the probability it is a spade. [Oct 2024]

Solution. $|S| = 52$, and there are 13 spades, so

$$p(\text{spade}) = \frac{13}{52} = \frac{1}{4}.$$

Worked Example — At Least One Head

Find the probability of obtaining at least one head when three coins are tossed. [Oct 2024]

Solution. From S above ($|S| = 8$), only TTT has *no* heads, so $|E| = 8 - 1 = 7$:

$$p(\text{at least one head}) = \frac{7}{8}.$$

Definition

If outcomes $a_1, \ldots, a_n$ of S are *not* equally likely, with (given or computed) probabilities $p(a_1), \ldots, p(a_n)$ summing to 1, then for an event $E = \{a_{i_1}, \ldots, a_{i_k}\}$,

$$p(E) = \sum_{i=1}^{k} p(a_i)$$

— the probability of an event is the sum of the probabilities of its individual outcomes.

Worked Example — Unequal Outcomes

Suppose the probability of rolling a prime number on a certain *loaded* die is twice that of rolling a non-prime, when the die is rolled once. Find the probability of rolling an odd number.
Solution. Six outcomes; three are prime $(2, 3, 5)$, three are non-prime $(1, 4, 6)$. Given $p(\text{prime}) = 2\,p(\text{nonprime})$, and the six individual probabilities must sum to 1:

$$\begin{aligned} 3\,p(\text{prime}) + 3\,p(\text{nonprime}) &= 1 \\ 6\,p(\text{nonprime}) + 3\,p(\text{nonprime}) &= 1 \\ p(\text{nonprime}) &= \tfrac{1}{9}. \end{aligned}$$

So $p(\text{prime}) = \frac{2}{9}$. The odd outcomes are $1, 3, 5$ (nonprime, prime, prime):

$$p(\text{odd}) = p(1) + p(3) + p(5) = \tfrac{1}{9} + \tfrac{2}{9} + \tfrac{2}{9} = \tfrac{5}{9}.$$

4.4.1 Mutually Exclusive Events

Definition

Two events $A, B \subseteq S$ are **mutually exclusive** if $A \cap B = \varnothing$ — they cannot occur simultaneously.

Worked Example

Drawing a red queen and drawing a black king from a deck of cards are mutually exclusive events (no card is both).

Theorem 6.20 — Inclusion–Exclusion Principle (for probability)

For any two events A, B of a finite sample space S,

$$p(A \cup B) = p(A) + p(B) - p(A \cap B).$$

(This follows immediately from $|A \cup B| = |A| + |B| - |A \cap B|$ divided by $|S|$; the proof is omitted here, per the syllabus.) In particular, if A and B are mutually exclusive, $A \cap B = \varnothing$ so $p(A \cap B) = 0$, giving the simpler

$$p(A \cup B) = p(A) + p(B) \qquad \text{(mutually exclusive events)}.$$

Worked Example — Red Queen or Black King

Find the probability of drawing a red queen or a black king from a standard deck.
Solution. These are mutually exclusive (2 red queens, 2 black kings, no overlap):

$$p(\text{red queen or black king}) = \frac{2}{52} + \frac{2}{52} = \frac{4}{52} = \frac{1}{13}.$$

Exam Tip

Section 6.8 is examinable through Example 6.49 (equally-likely and non-equally-likely probability computations), plus mutually exclusive events and Theorem 6.20's *statement* (not its proof). Classical probability $p(E) = |E|/|S|$ — where $|E|$ is often itself a permutation or combination count — is the single most tested idea of this whole chapter.

Module V Extension — Conditional Probability and Beyond

Module V Extension — Not Examined

Conditional probability. The probability of A *given that* B has already occurred is

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)}, \qquad p(B) \neq 0.$$

Intuitively, once we know B happened, B becomes the new (smaller) sample space.

Multiplication theorem of probability. Rearranging the definition above,

$$p(A \cap B) = p(B) \cdot p(A \mid B) = p(A) \cdot p(B \mid A).$$

Independent vs. dependent events. Events A, B are **independent** if the occurrence of one does not affect the probability of the other: $p(A \mid B) = p(A)$, equivalently $p(A \cap B) = p(A)\, p(B)$. If this fails, A and B are **dependent**.
Example. Draw two cards *with replacement*: the draws are independent, and $p(\text{both aces}) = \frac{4}{52} \cdot \frac{4}{52}$. Draw two cards *without replacement*: the draws are dependent, and $p(\text{both aces}) = \frac{4}{52} \cdot \frac{3}{51}$ — the second probability changes because the first card is not put back.

Probability distributions. A **random variable** X assigns a number to each outcome of an experiment (e.g., X = number of heads in three coin tosses). Its **probability distribution** lists every possible value of X with its probability; e.g., for three fair coin tosses, $X \in \{0, 1, 2, 3\}$ with $p(X{=}0) = \frac{1}{8}$, $p(X{=}1) = \frac{3}{8}$, $p(X{=}2) = \frac{3}{8}$, $p(X{=}3) = \frac{1}{8}$ (from $\binom{3}{k}/2^3$, using the combination formula of Section 4.3!). The **expected value** $E[X] = \sum x\, p(X{=}x)$ is the long-run average.

Correlation and regression. Given paired data (x_i, y_i), the **correlation coefficient** r (between -1 and 1) measures how strongly x and y move together linearly; r near ± 1 means a strong linear relationship, r near 0 means little to none. **Linear regression** fits the "best" straight line $\hat{y} = a + bx$ through the data (minimising total squared vertical distance) — the standard tool for predicting y from x.
This entire extension is Module V, open-ended material and **not** externally examined in MAT1MN104.

Chapter Summary

- **Addition principle** (mutually exclusive tasks: add), **inclusion–exclusion for tasks** ($m + n - k$), **multiplication principle** (sequential subtasks: multiply).
- **Permutations**: $P(n, r) = \dfrac{n!}{(n-r)!}$ (order matters); **cyclic permutations** of n items: $(n-1)!$.
- **Combinations**: $C(n, r) = \dfrac{n!}{r!(n-r)!}$ (order does not matter).
- **Probability**: $p(E) = |E|/|S|$ for equally likely outcomes, or $p(E) = \sum p(a_i)$ otherwise; **mutually exclusive events** ($A \cap B = \varnothing$) satisfy $p(A \cup B) = p(A) + p(B)$; in general $p(A \cup B) = p(A) + p(B) - p(A \cap B)$.
- *Module V extension:* conditional probability, independence, probability distributions, correlation/regression.

Practice Problems

1. A restaurant offers 4 starters, 6 mains, 3 desserts. How many different 3-course meals are possible (multiplication principle)?
2. A student may take one elective from 8 Mathematics electives or 5 Computer Science electives (no overlap). In how many ways can the elective be chosen?
3. Compute $P(7, 3)$ and $C(7, 3)$. Explain, in words, the difference between what each one counts.
4. In how many ways can 6 people be seated (i) in a row, (ii) around a circular table?
5. Find the number of distinct arrangements of the letters of the word "LOGIC".
6. A committee of 4 is to be chosen from 6 men and 5 women. In how many ways can this be done if the committee must have exactly 2 men and 2 women?
7. A single fair die is rolled. Find the probability of rolling a number greater than 4.
8. Two dice are rolled. Find the probability that the sum is 7. (*Hint*: $|S| = 36$; list the favorable outcomes.)
9. Are the events "sum is 7" and "first die shows 6" (when rolling two dice) mutually exclusive? Compute p(sum is 7 or first die shows 6).
10. A bag has 5 red and 3 blue marbles. Find the probability of drawing a red marble or a marble numbered... (design your own mutually-exclusive-events probability problem modelled on the red-queen-or-black-king example, and solve it).
11. (Module V) Two cards are drawn without replacement from a standard deck. Find the probability both are hearts, using the multiplication theorem of conditional probability.
12. Find the number of 5-card poker hands (from a standard 52-card deck) that contain exactly 3 aces. (*Hint*: choose the 3 aces, then choose the remaining 2 cards from the non-aces — two combinations, multiplied.)

Quick Reference: Key Formulas and Laws

The table below collects the formulas and laws used most often across Modules I–IV. Use this for quick revision before the exam.

Item	**Module**	**Statement**
De Morgan's Laws (logic)	I	$\sim(p \vee q) \equiv \sim p \wedge \sim q$; $\quad \sim(p \wedge q) \equiv \sim p \vee \sim q$
Contrapositive	I	$p \to q \equiv \sim q \to \sim p$
De Morgan's Laws (sets)	II	$(A \cup B)' = A' \cap B'$; $\quad (A \cap B)' = A' \cup B'$
Cardinality of a union	II	$\lvert A \cup B\rvert = \lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$
Inclusion–exclusion (3 sets)	II	$\lvert A \cup B \cup C\rvert = \lvert A\rvert + \lvert B\rvert + \lvert C\rvert - \lvert A \cap B\rvert - \lvert A \cap C\rvert - \lvert B \cap C\rvert + \lvert A \cap B \cap C\rvert$
Cartesian product size	II	$\lvert A \times B\rvert = \lvert A\rvert \cdot \lvert B\rvert$
Floor/ceiling	III	$\lfloor x \rfloor$, $\lceil x \rceil$; $\lceil x \rceil = \lfloor x \rfloor + 1$ if $x \notin \mathbb{Z}$
Matrix sizes	III	$A_{m\times n}$; AB defined only if (cols of A) = (rows of B)
Fundamental counting principle	IV	sequential choices: multiply; alternative choices: add
r-permutations	IV	$P(n,r) = \dfrac{n!}{(n-r)!}$
r-combinations	IV	$C(n,r) = \dfrac{n!}{r!(n-r)!}$
Classical probability	IV	$P(E) = \dfrac{\lvert E\rvert}{\lvert S\rvert}$
Mutually exclusive events	IV	$P(E \cup F) = P(E) + P(F)$

Hints and Answers to Practice Problems

This section gives a hint or a final answer for every practice problem in Modules I–IV — not a full worked solution. For computational problems, the final answer is given so you can check your own working. For questions that ask for a definition, explanation, or short answer, a compact model answer is given. Since proofs are not part of the external exam for this course, proof-style problems get a hint naming the key laws to chain together, not the full write-up. *Attempt every problem on your own first.* Use this section to check your work, not to skip the work.

Module I — Mathematical Logic

1. See the truth tables for $\wedge$ and $\vee$ in Section 1.1.
2. By De Morgan's law: "The exam is not easy or the hall is not cool" ($\sim p \vee \sim q$).
3. $(\forall x)(\exists y)(x + y = 0)$.
4. A contradiction is false for every combination of truth values, e.g. $p \wedge \sim p$. A different example: $(p \to q) \wedge (p \wedge \sim q)$ — asserting an implication and its own violation simultaneously.
5. The table for $(p \to q) \wedge (q \to p)$ is identical to the table for $p \leftrightarrow q$: T,F,F,T. So $(p \to q) \wedge (q \to p) \equiv p \leftrightarrow q$.
6. Converse: "If a number is divisible by 3, it is divisible by 6" — *false* (e.g. 9). Inverse: "If not divisible by 6, then not divisible by 3" — also false (same counterexample, since inverse $\equiv$ converse). Contrapositive: "If not divisible by 3, then not divisible by 6" — *guaranteed true*, since it is equivalent to the (true) original statement.
7. $(p \wedge q) \vee (p \wedge \sim q) \equiv p \wedge (q \vee \sim q)$ [distributive] $\equiv p \wedge t$ [inverse law] $\equiv p$ [identity law].
8. Build the 4-row table for $\sim (p \vee q)$ and for $\sim p \wedge \sim q$; both read F,F,F,T — identical, confirming the equivalence.
9. Symbolically: $(\forall x)(Sx \to Px)$ and $(\exists x)(Sx \wedge \sim Px)$, where Sx: "x studies", Px: "x passes". The second is exactly $\sim (\forall x)(Sx \to Px)$ — the negation of the first. They **cannot** both be true; the pair is inconsistent.
10. Let p: it rains, q: match cancelled. Premises $p \to q$, $\sim p$; conclusion $\sim q$. This is the **fallacy of denying the antecedent** — invalid ($\sim p$ and $p \to q$ do not let you conclude $\sim q$).
11. $p \to (q \to r) \equiv \sim p \vee (q \to r)$ [impl. conversion] $\equiv \sim p \vee (\sim q \vee r)$ [impl. conversion] $\equiv (\sim p \vee \sim q) \vee r$ [associative] $\equiv \sim (p \wedge q) \vee r$ [De Morgan] $\equiv (p \wedge q) \to r$ [impl. conversion].
12. Chithra's statement is logically the negation of Bala's ($C \equiv \sim B$), so exactly one

of $\{B, C\}$ is automatically true regardless of the facts. Since exactly one of *all three* statements is true, Asha's statement must be **false** — so Bala did not eat the mango. Checking the remaining two cases (Chithra guilty vs. Asha guilty) against "exactly one true statement" shows **both are self-consistent**, while "Bala guilty" produces two true statements (a contradiction) and is ruled out. This puzzle, as stated, pins down only that *Bala is innocent*; a fully unique culprit needs one more clue — a useful reminder that logic tells you exactly what your premises do and do not determine.

Module II — Set Theory

1. $\{x \mid x \text{ is a letter in "MALAYALAM"}\} = \{M, A, L, Y\}$ (4 distinct letters).
2. $A \cup B = \{a, b, c, d, e, x, y, z\}$, so $(A \cup B)' = \{w\}$. $A' = \{c, d, e, w\}$, $B' = \{a, b, w\}$, so $A' \cap B' = \{w\}$. They match — De Morgan's law, $(A \cup B)' = A' \cap B'$.
3. $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ — $8 = 2^3$ elements.
4. $$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$
   $$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$
   Not equal as sets (different ordered pairs), though $|A \times B| = |B \times A| = 6$.
5. $(A \cap B') \cup (A' \cap B') = (A \cup A') \cap B' = U \cap B' = B'$ [distributive, inverse]. So the expression becomes $B' \cup (A \cap B) = (B' \cup A) \cap (B' \cup B) = (B' \cup A) \cap U = A \cup B'$ [distributive, inverse, identity].
6. $|A \cup B| = 3 + 5 - 2 = 6$; $|A \times B| = |A| \cdot |B| = 15$.
7. Answers will vary; check your final count against $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$, and "none of the three" $= 100 - |A \cup B \cup C|$.
8. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$, so in particular $x \in A$. Hence $A \cap B \subseteq A$.
9. $\lfloor 2000/5 \rfloor = 400$, $\lfloor 2000/9 \rfloor = 222$, $\lfloor 2000/45 \rfloor = 44$. $|A \cup B| = 400 + 222 - 44 = 578$. Not divisible by either: $2000 - 578 = 1422$.
10. E.g. $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (2, 1), (3, 4), (4, 3)\}$: symmetric (every pair reversed), not reflexive (no self-loops). Digraph: two double arrows, $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, no loops.
11. E.g. $A = \{1, 2, 3\}$, $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ (the $\leq$ relation): reflexive and transitive, but $(1, 2) \in R$ while $(2, 1) \notin R$ — fails symmetry.
12. $A - (B \cup C) = A \cap (B \cup C)' = A \cap (B' \cap C')$ [De Morgan] $= (A \cap B') \cap (A \cap C')$ [rearrange, using $A = A \cap A$] $= (A - B) \cap (A - C)$.

Module III — Functions and Matrices

1. (i) is a function. (ii) is not (element 3 unassigned). (iii) is not (element 1 has two images).
2. $(f + g)(x) = x^2 - 1 + \sqrt{x}$, $(fg)(x) = (x^2 - 1)\sqrt{x}$; domain $= [0, \infty)$ (intersection of $\mathrm{Dom}(f) = \mathbb{R}$ and $\mathrm{Dom}(g) = [0, \infty)$).
3. $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$; graph is a "V" with vertex at the origin.
4. $\lfloor -2.3 \rfloor = -3$, $\lceil -2.3 \rceil = -2$, $\lfloor 7 \rfloor = 7$, $17 \bmod 5 = 2$, $17 \operatorname{div} 5 = 3$.

5. $f_A(3) = 1$, $f_B(3) = 1$, so $f_A(3)f_B(3) = 1$; $A \cap B = \{3, 5, 7\}$, so $f_{A\cap B}(3) = 1$. ✓ $f_A(4) = 0$, $f_B(4) = 0$, product 0; $4 \notin A \cap B$, so $f_{A\cap B}(4) = 0$. ✓
6. 250 mod 7 = 5 (since $250 = 35 \times 7 + 5$). Five days after Wednesday: Thu, Fri, Sat, Sun, **Monday**.
7. $A + B = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$, $3A = \begin{bmatrix} 6 & 0 \\ -3 & 9 \end{bmatrix}$, $AB = \begin{bmatrix} 2 & 8 \\ 5 & -10 \end{bmatrix}$, $BA = \begin{bmatrix} -2 & 12 \\ 6 & -6 \end{bmatrix}$. $AB \neq BA$.
8. A is 2×3, B is 3×4: AB is defined, size 2×4. BA: columns of B $(4) \neq$ rows of A (2), so BA is **not** defined.
9. $-A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$, so $A + (-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$. ✓
10. $f(1) = -2$, $f(2) = 4$ (opposite signs, root in $[1, 2]$). Midpoint $c = 1.5$: $f(1.5) = -0.125$, same sign as $f(1)$, so the root is narrowed to $[1.5, 2]$.
11. $x = 2$, $y = 1$. (Check: $2(2) + 1 = 5$ ✓, $2 - 1 = 1$ ✓.)
12. Setting $A = O$ (the zero matrix, itself an $m \times n$ matrix) in $A + B = A$ gives $O + B = O$, and $O + B = B$ by the identity law, so $B = O$.

Module IV — Combinatorics and Discrete Probability

1. $4 \times 6 \times 3 = 72$ meals.
2. $8 + 5 = 13$ ways (mutually exclusive choices).
3. $P(7, 3) = 7!/4! = 210$; $C(7, 3) = 7!/(3!\,4!) = 35$. $P(7, 3)$ counts *ordered* arrangements of 3 items from 7; $C(7, 3)$ counts *unordered* selections — indeed $P(7, 3) = C(7, 3) \times 3!$.
4. Row: $6! = 720$. Circle: $(6 - 1)! = 120$.
5. LOGIC has 5 distinct letters: $5! = 120$ arrangements.
6. $C(6, 2) \times C(5, 2) = 15 \times 10 = 150$.
7. Favourable outcomes $\{5, 6\}$: $p = 2/6 = 1/3$.
8. Favourable outcomes summing to 7: $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$ — 6 outcomes out of $|S| = 36$: $p = 6/36 = 1/6$.
9. Not mutually exclusive — $(6, 1)$ lies in both events. $p(\text{sum } 7) = 6/36$, $p(\text{first} = 6) = 6/36$, $p(\text{both}) = 1/36$; so $p(\text{union}) = 6/36 + 6/36 - 1/36 = 11/36$.
10. Answers will vary; check that mutually exclusive events satisfy $p(A \cup B) = p(A) + p(B)$ exactly, with no overlap term to subtract.
11. $p(\text{both hearts}) = \dfrac{13}{52} \cdot \dfrac{12}{51} = \dfrac{156}{2652} = \dfrac{1}{17}$.
12. $C(4, 3) \times C(48, 2) = 4 \times 1128 = 4512$ hands.

Index

r-combination, 33
r-permutation, 32

argument, 8
augmented matrix, 27

biconditional, 4

cardinality, 13
Cartesian product, 16
ceiling, 22
characteristic function, 23
codomain, 20
column vector, 24
Complement, 14
complement, 35
conclusion, 3, 8
conditional, 3
conjunction, 2
contains, 12
contingency, 5
contradiction, 5
Contrapositive, 3
Converse, 3
correlation coefficient, 37

dependent, 37
Difference, 14
digraph, 18
directed graph, 18
disjoint, 14
disjunction, 2
div function, 23
domain, 20
double subscript notation, 24

element, 24
elementary row operations, 27
elements, 11
empty, 12
entry, 24
equal, 12, 21, 24
equivalence relation, 18
event, 34
existential quantifier, 7
expected value, 37
exponential function, 22

fallacy of affirming the consequent, 9
favorable, 35
finite, 13
floor, 22
function, 20

Gauss–Jordan method, 27

hypothesis, 3

identity matrix, 24
image, 20
implication, 3
impossible event, 35
independent, 37
inference rules, 9
infinite, 13
Intersection, 14
invalid, 8
Inverse, 3

linear, 22
Linear regression, 37
logarithm, 22
logically equivalent, 5

main diagonal, 24
matrix, 24
members, 11
mod function, 23
mutually exclusive, 36

negation, 2
negative, 25
null, 12

on, 18

permutation, 31

piecewise, 21
polynomial function, 22
power set, 12
pre-image, 20
predicate, 7
premises, 8
principle of inclusion–exclusion, 17
probability, 35
probability distribution, 37
product, 21, 25
proper subset, 12
proposition, 1

quadratic, 22

random variable, 37
reduced row-echelon form, 27
reflexive, 18
relation, 18
root, 27
row vector, 24

sample space, 34
set, 11, 12
simple event, 35
size, 24
square matrix of order n, 24
statement, 1
subset, 12
sum, 21
symmetric, 18
Symmetric difference, 14

tautology, 5
transitive, 18
truth value, 1

unfavorable, 35
Union, 14
universal quantifier, 7
universal set, 12

valid, 8
value, 20
Venn diagram, 13

zero matrix, 24