CALCULUS

Lecture 15: Partial Differential Equations

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Table of Contents

ODEs vs PDEs

Ordinary DE (ODE)

- One independent variable
- Derivatives are ordinary: dy/dx

Examples:

$$x^2 y' = \sin x$$

$$\frac{d^2 y}{du^2} + u^2 y = \sin u$$

Partial DE (PDE)

- Two or more independent variables
- Derivatives are partial: $\partial u/\partial t$

Example:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + xu = \sin x$$

PDEs arise naturally in physics and engineering:

- Heat conduction: temperature depends on both position x and time t
- Wave propagation: displacement depends on x and t
- Electrostatics: potential depends on position (x, y, z)

Notation: Partial Derivatives

For $u = u(x,t)$ or $u = u(x,y)$, we use a compact subscript notation:

Subscript notation

$$u_x = \frac{\partial u}{\partial x}$$

$$u_t = \frac{\partial u}{\partial t}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xt} = \frac{\partial^2 u}{\partial x\, \partial t}$$

Note on mixed partials

For continuous functions:

$$u_{xt} = u_{tx}$$

(order of differentiation does not matter)

Example: $u = x^2 t^3$

$$u_x = 2xt^3$$
$$u_{xt} = 6xt^2 = u_{tx}$$

Order and Linearity of PDEs

Order: determined by the highest-order partial derivative.

- $u_x + u_t = 0$ (1st order)
- $u_{xx} + u_t = 0$ (2nd order)
- $u_{xxx} + u_y = 0$ (3rd order)

Linearity: A PDE is **linear** if u and all its derivatives appear to the first power and are not multiplied together.

- **Linear:** $u_{xx} + u_{yy} = 0$ (Laplace)
- **Non-linear:** $u \cdot u_x + u_y = 0$ (u multiplied by u_x)
- **Non-linear:** $u_x^2 + u_y = 0$ (u_x squared)

Homogeneous vs non-homogeneous:

- **Homogeneous:** every term contains u or its derivatives (no standalone function of the independent variables)
- **Non-homogeneous:** has a forcing term, e.g. $u_{xx} + u_t = \sin x$

General Solution Contains Arbitrary Functions

A key difference between ODEs and PDEs:

ODE general solution
Contains arbitrary *constants*:

$$y'' = 0 \implies y = c_1 + c_2 x$$

ICs fix c_1, c_2.

PDE general solution
Contains arbitrary *functions*:

$$Z_x = 0 \implies Z = f(y)$$

f can be any function of y.

Why? Integrating w.r.t. x when y is present: the "constant" can depend on y.

$$\int Z_x \, dx = \int 0 \, dx \implies Z = f(y)$$

Boundary conditions are needed to pin down the arbitrary functions.

Table of Contents

Solving $u_x = 0$: Direct Integration

Problem: Solve $u_x = 0$ given $u(0,t) = t^2$.

Step 1: Integrate w.r.t. x (treating t as a parameter):

$$\int u_x \, dx = \int 0 \, dx \implies u(x,t) = f(t)$$

The "constant" of integration is an arbitrary function of t.

Step 2: Apply the boundary condition $u(0,t) = t^2$:

$$f(t) = t^2$$

Solution:

$$u(x,t) = t^2$$

(The function is constant in x but varies in t.)

Solving $Z_{xy} = 0$: Two Integrations

Problem: Solve $Z_{xy} = 0$ for $Z = Z(x, y)$.

Step 1: Integrate w.r.t. y (treating x as constant):

$$\int Z_{xy}\, dy = \int 0\, dy \implies Z_x = g(x)$$

The "constant" is an arbitrary function $g(x)$.

Step 2: Integrate w.r.t. x:

$$\int Z_x\, dx = \int g(x)\, dx \implies Z = G(x) + h(y)$$

where $G(x)$ is any antiderivative of $g(x)$ (another arbitrary function of x).

General Solution

$Z(x, y) = f(x) + h(y)$ where f and h are arbitrary functions.

Direct Integration with Boundary Conditions

Problem: Solve $Z_{xy} = x + y$, given $Z(x,0) = x^2$ and $Z(0,y) = y^3$.

Step 1: Integrate w.r.t. y: $Z_x = xy + \frac{y^2}{2} + f(x)$

Step 2: Integrate w.r.t. x:

$$Z = \frac{x^2 y}{2} + \frac{xy^2}{2} + F(x) + g(y), \quad F'(x) = f(x)$$

Apply BCs:

- $Z(x,0) = x^2$: $F(x) = x^2$ (taking $g(0) = 0$)
- $Z(0,y) = y^3$: $g(y) = y^3$

$$Z(x,y) = \frac{x^2 y}{2} + \frac{xy^2}{2} + x^2 + y^3$$

Practice: Direct Integration

Problem 1: Solve $u_t = 0$ given $u(x,0) = \sin x$.

Solution: Integrate w.r.t. t: $u = f(x)$. Apply IC: $f(x) = \sin x$.

$$u(x,t) = \sin x$$

Problem 2: Solve $Z_{xy} = 2x$ given $Z(x,0) = x^2$ and $Z(0,y) = 0$.

Solution: Integrate w.r.t. y: $Z_x = 2xy + f(x)$.
Integrate w.r.t. x: $Z = x^2 y + F(x) + g(y)$.
$Z(x,0) = x^2$: $F(x) = x^2$. $Z(0,y) = 0$: $g(y) = 0$.

$$Z(x,y) = x^2 y + x^2 = x^2(y+1)$$

Table of Contents

General Linear Second-Order PDE

The general form of a **linear second-order PDE** in two independent variables x and y (or x and t) is:

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + Fu = G$$

where A, B, C, D, E, F, G may be constants or functions of x and y.

- If $G = 0$: **homogeneous**
- If $G \neq 0$: **non-homogeneous**

Classification is based on the **discriminant** $B^2 - 4AC$, analogous to the discriminant of a conic section $Ax^2 + Bxy + Cy^2 + \cdots$:

$$B^2 - 4AC > 0 \implies \textbf{Hyperbolic}$$
$$B^2 - 4AC = 0 \implies \textbf{Parabolic}$$
$$B^2 - 4AC < 0 \implies \textbf{Elliptic}$$

Classification Examples

Example 1: Parabolic

$$3\,u_{xx} + u_y = 0 \implies A = 3,\ B = 0,\ C = 0$$

$B^2 - 4AC = 0 - 0 = 0 \implies$ **Parabolic**

Example 2: Hyperbolic

$$u_{xx} - u_{yy} = 0 \implies A = 1,\ B = 0,\ C = -1$$

$B^2 - 4AC = 0 - 4(1)(-1) = 4 > 0 \implies$ **Hyperbolic**

Example 3: Elliptic

$$u_{xx} + u_{yy} = 0 \implies A = 1,\ B = 0,\ C = 1$$

$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0 \implies$ **Elliptic**

Practice: Classify the PDE

Classify each PDE using the discriminant $B^2 - 4AC$.

(a) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$
$A = 1,\ B = 4,\ C = 4$: $B^2 - 4AC = 16 - 16 = 0 \implies$ **Parabolic**

(b) $u_{xx} - 2u_{xy} + 3u_{yy} = 0$
$A = 1,\ B = -2,\ C = 3$: $B^2 - 4AC = 4 - 12 = -8 < 0 \implies$ **Elliptic**

(c) $u_{xx} + 6u_{xy} - 7u_{yy} = 0$
$A = 1,\ B = 6,\ C = -7$: $B^2 - 4AC = 36 + 28 = 64 > 0 \implies$ **Hyperbolic**

Table of Contents

Initial and Boundary Value Problems

Initial Conditions (ICs): specify the state at $t = 0$ over all space.

- $u(x, 0) = f(x)$ (initial displacement)
- $u_t(x, 0) = g(x)$ (initial velocity)

Boundary Conditions (BCs): specify the state at spatial boundaries for all time.

- $u(0, t) = 0, \quad u(L, t) = 0$ (Dirichlet BCs: fixed endpoints)
- $u_x(0, t) = 0$ (Neumann BC: insulated/free endpoint)

Wave equation formulation: A vibrating string of length L:

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \ t > 0$$

with BCs $u(0, t) = u(L, t) = 0$ and ICs $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$.

Lagrange's Method for First-Order PDEs

A **linear first-order PDE** of the form $Pp+Qq = R$, where $p = z_x$, $q = z_y$:

$$P(x, y, z)\,\frac{\partial z}{\partial x} + Q(x, y, z)\,\frac{\partial z}{\partial y} = R(x, y, z)$$

Lagrange's Method: Auxiliary Equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Solve these symmetric ODEs to find two independent solutions $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$. The general solution is $F(u_1, u_2) = 0$ (or $u_1 = \phi(u_2)$) for any function F (or ϕ).

Lagrange's Method: Worked Example

Problem: Solve $x\,z_x + y\,z_y = z$.
This is $Pp + Qq = R$ with $P = x$, $Q = y$, $R = z$.
Auxiliary equations:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

From $dx/x = dy/y$:

$$\int \frac{dx}{x} = \int \frac{dy}{y} \implies \ln|x| = \ln|y| + \ln|c_1| \implies \frac{x}{y} = c_1$$

From $dx/x = dz/z$:

$$\ln|x| = \ln|z| + \ln|c_2| \implies \frac{x}{z} = c_2$$

General Solution

$$F\left(\frac{x}{y}, \frac{x}{z}\right) = 0 \quad \text{or equivalently} \quad z = x\,\phi\left(\frac{x}{y}\right)$$

Practice: Lagrange's Method

Problem: Solve $y^2\, z_x - xy\, z_y = x(z - 2y)$.

Here $P = y^2$, $Q = -xy$, $R = x(z - 2y)$.
Auxiliary equations:

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

From $dx/y^2 = dy/(-xy)$:

$$-xy\, dx = y^2\, dy \implies -x\, dx = y\, dy$$
$$\implies -\tfrac{x^2}{2} = \tfrac{y^2}{2} + c \implies x^2 + y^2 = c_1$$

Second solution: Use $dx/y^2 = dy/(-xy)$; integrate to confirm c_1.

(Finding the second independent solution c_2 requires substituting back and solving a related ODE.)
General form: $F(x^2 + y^2,\ c_2) = 0$.

Table of Contents

The Separation Assumption

Key idea: Assume the solution factors into a product of functions, each depending on one variable only:

$$u(x, y) = X(x) \cdot Y(y)$$

Why does this work? Substituting into the PDE separates it into two ODEs — one in x, one in y — connected by a constant called the **separation constant**.

Working PDE: $u_{xx} = 4u_y$
Compute: $u_{xx} = X''Y, \quad u_y = XY'$.
Substituting:

$$X''Y = 4XY' \implies \frac{X''}{4X} = \frac{Y'}{Y}$$

Left side depends only on x; right side only on y. **Both must equal the same constant $-\lambda$.**

The Two ODEs and Separation Constant

Setting $\dfrac{X''}{4X} = \dfrac{Y'}{Y} = -\lambda$, we get two ODEs:

$$X'' + 4\lambda X = 0 \qquad \text{and} \qquad Y' + \lambda Y = 0$$

The value of λ determines the type of solution. We consider three cases:

Case	λ	Solutions
1	$\lambda = 0$	Linear / constant
2	$\lambda = \alpha^2 > 0$	Oscillatory in x, decay in y
3	$\lambda = -\alpha^2 < 0$	Exponential in x, growth in y

The physically meaningful case depends on the problem. For the **heat equation** (bounded domain), Case 2 ($\lambda > 0$) gives the decaying, physically relevant solutions.

Case 1: $\lambda = 0$

The ODEs become $X'' = 0$ and $Y' = 0$.

For X:

$$X'' = 0 \implies X' = c_2 \implies X = c_1 + c_2 x$$

For Y:

$$Y' = 0 \implies Y = c_3$$

Product solution:

$$u(x, y) = (c_1 + c_2 x) \cdot c_3$$

Letting $A = c_1 c_3$ and $B = c_2 c_3$:

$$u(x, y) = A + Bx$$

A linear function in x, constant in y.

CASE 2: $\lambda = \alpha^2 > 0$

For X: $X'' + 4\alpha^2 X = 0$

$$m^2 + 4\alpha^2 = 0 \implies m = \pm 2\alpha i$$
$$X(x) = c_1 \cos(2\alpha x) + c_2 \sin(2\alpha x)$$

For Y: $Y' + \alpha^2 Y = 0$

$$m + \alpha^2 = 0 \implies m = -\alpha^2$$
$$Y(y) = c_3\, e^{-\alpha^2 y}$$

Product solution:

$$u(x, y) = \big(c_1 \cos(2\alpha x) + c_2 \sin(2\alpha x)\big)\, c_3\, e^{-\alpha^2 y}$$

The solution oscillates in x and decays exponentially in y — physically the most important case for bounded domains.

Case 3: $\lambda = -\alpha^2 < 0$

For X: $X'' - 4\alpha^2 X = 0$

$$m^2 - 4\alpha^2 = 0 \implies m = \pm 2\alpha$$
$$X(x) = c_1 e^{2\alpha x} + c_2 e^{-2\alpha x}$$

For Y: $Y' - \alpha^2 Y = 0$

$$m - \alpha^2 = 0 \implies m = \alpha^2$$
$$Y(y) = c_3\, e^{\alpha^2 y}$$

Product solution (absorbing c_3: let $A_1 = c_1 c_3$, $B_1 = c_2 c_3$):

$$u(x, y) = A_1\, e^{2\alpha x + \alpha^2 y} + B_1\, e^{-2\alpha x + \alpha^2 y}$$

The solution grows exponentially in y — rarely physical, but mathematically valid.

Table of Contents

The Heat Equation — Physical Derivation

Consider temperatures T_1, T_2, T_3 at adjacent points x_1, x_2, x_3. The rate of change of T_2 is proportional to the *average of its neighbours* minus itself:

$$\frac{dT_2}{dt} \propto \frac{T_1 + T_3}{2} - T_2 = \frac{T_1 + T_3 - 2T_2}{2} = \frac{(T_3 - T_2) - (T_2 - T_1)}{2}$$

This is the discrete second derivative. In the continuous limit:

Heat Equation (1D)

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad u_t = k\,u_{xx}$$

In 3D: $u_t = k(u_{xx} + u_{yy} + u_{zz})$
Here $k > 0$ is the **thermal diffusivity** of the material.

The Wave Equation and Laplace's Equation

Wave Equation:

$$\alpha^2 u_{xx} = u_{tt} \quad \text{or} \quad u_{tt} = \alpha^2 u_{xx}$$

Models vibrating strings, sound propagation, and electromagnetic waves. α is the wave speed.

Laplace's Equation:

$$u_{xx} + u_{yy} = 0$$

Models **steady-state** (equilibrium) situations — temperature distributions where $u_t = 0$, electrostatic potentials, and fluid flow.
Note: Laplace's equation is the heat equation with $u_t = 0$ (no time dependence).

Classification of the Three Fundamental PDEs

Apply $B^2 - 4AC$ to each (let second pair of variables be x, t or x, y):

Heat Equation $ku_{xx} - u_t = 0$: $\quad A = k$, $B = 0$, $C = 0$

$$B^2 - 4AC = 0 \implies \textbf{Parabolic}$$

Wave Equation $\alpha^2 u_{xx} - u_{tt} = 0$: $\quad A = \alpha^2$, $B = 0$, $C = -1$

$$B^2 - 4AC = 4\alpha^2 > 0 \implies \textbf{Hyperbolic}$$

Laplace's Equation $u_{xx} + u_{yy} = 0$: $\quad A = 1$, $B = 0$, $C = 1$

$$B^2 - 4AC = -4 < 0 \implies \textbf{Elliptic}$$

Parabolic = diffusion/conduction *Hyperbolic = propagation*
Elliptic = equilibrium

Second-Order PDE Solved as an ODE

Problem: Solve $u_{xx} - u = 0$ where $u = u(x, t)$.

Since there are *no* t-derivatives, treat this as an ODE in x alone (with t as a parameter):

$$m^2 - 1 = 0 \implies m = \pm 1$$

The "constants" of integration can be *arbitrary functions of* t:

General Solution

$$u(x, t) = c_1(t)\, e^x + c_2(t)\, e^{-x}$$

Verify:

$$\begin{aligned} u_x &= c_1(t)e^x - c_2(t)e^{-x} \\ u_{xx} &= c_1(t)e^x + c_2(t)e^{-x} = u \quad \checkmark \end{aligned}$$

Practice: Separation of Variables

Problem: Find all separable solutions of $u_{xx} = 9u_y$.

Classify: $A = 1,\ B = 0,\ C = 0 \implies B^2 - 4AC = 0 \implies$ **Parabolic**

Solution: Let $u = X(x)Y(y)$: $X''Y = 9XY' \implies X''/9X = Y'/Y = -\lambda$.

Two ODEs: $X'' + 9\lambda X = 0$ and $Y' + \lambda Y = 0$.

- **Case $\lambda = 0$:** $u = A + Bx$
- **Case $\lambda = \alpha^2 > 0$:**

$$X = c_1\cos(3\alpha x) + c_2\sin(3\alpha x), \quad Y = c_3 e^{-\alpha^2 y}$$

$$u = \big(c_1\cos(3\alpha x) + c_2\sin(3\alpha x)\big)e^{-\alpha^2 y}$$

- **Case $\lambda = -\alpha^2 < 0$:**

$$u = A_1 e^{3\alpha x + \alpha^2 y} + B_1 e^{-3\alpha x + \alpha^2 y}$$

Table of Contents

Practice 1: Classify and Solve by Integration

Problem: Classify and solve $u_{xt} = e^{x+t}$, given $u(x,0) = e^x$ and $u(0,t) = e^t$.

Classification: This is first-order in x and t jointly (mixed partial) — not covered by the $B^2 - 4AC$ rule for second-order equations. Solve directly.

Solution: Integrate w.r.t. t:

$$u_x = e^{x+t} + f(x)$$

Integrate w.r.t. x:

$$u = e^{x+t} + F(x) + g(t)$$

Apply $u(x,0) = e^x$: $e^x + F(x) + g(0) = e^x \implies F(x) = 0$.
Apply $u(0,t) = e^t$: $e^t + g(t) = e^t \implies g(t) = 0$.

$$\boxed{u(x,t) = e^{x+t}}$$

Practice 2: Full Classification Problems

Classify each of the following PDEs:

(a) $u_{xx} + 2u_{xy} + u_{yy} + u_x = 0$
$A = 1,\ B = 2,\ C = 1{:}\ \ B^2 - 4AC = 4 - 4 = 0 \implies$ **Parabolic**

(b) $4u_{xx} - u_{yy} + 3u_y = 0$
$A = 4,\ B = 0,\ C = -1{:}\ \ B^2 - 4AC = 0 + 16 = 16 > 0 \implies$
Hyperbolic

(c) $u_{xx} + 4u_{xy} + 5u_{yy} - u = 0$
$A = 1,\ B = 4,\ C = 5{:}\ \ B^2 - 4AC = 16 - 20 = -4 < 0 \implies$
Elliptic

Practice 3: Separation of Variables

Problem: Solve $u_{xx} = 4u_y$ with BCs $u(0, y) = 0$, $u(\pi, y) = 0$.
From earlier: Case 2 ($\lambda = \alpha^2 > 0$) gives:

$$X(x) = c_1 \cos(2\alpha x) + c_2 \sin(2\alpha x)$$

Apply $X(0) = 0$: $c_1 = 0$. So $X = c_2 \sin(2\alpha x)$.
Apply $X(\pi) = 0$ (non-trivial): $\sin(2\alpha\pi) = 0 \implies 2\alpha\pi = n\pi \implies \alpha = n/2$
For $n = 1, 2, 3, \ldots$: $X_n = \sin(nx)$, $Y_n = e^{-n^2 y/4}$

General Solution (Superposition)

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(nx)\, e^{-n^2 y/4}$$

The constants A_n are determined by the initial condition $u(x, 0)$ using Fourier series.

Thank You