CALCULUS

Lecture 13: Advanced Laplace Transforms

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Recall: Standard Transform Table

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$1/s$
t^n	$n!/s^{n+1}$
e^{at}	$1/(s-a)$
$\sin(kt)$	$k/(s^2+k^2)$
$\cos(kt)$	$s/(s^2+k^2)$

This lecture adds two powerful tools:

- **First Translation Rule:** handles $e^{at}f(t)$ by shifting $s \to s-a$
- **Second Translation Rule:** handles $f(t-a)u(t-a)$ by a factor e^{-as}

First Translation (s-Shifting) Rule

Theorem (First Translation Rule)

If $\mathcal{L}\{f(t)\} = F(s)$, then for any constant a:

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Proof:

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t)\, e^{-st}\, dt = \int_0^\infty f(t)\, e^{-(s-a)t}\, dt = F(s-a)$$

Key idea: multiplying by e^{at} in the time domain is equivalent to **replacing s with $s-a$** in the s-domain.

Mnemonic: "exponential times a function $\Leftrightarrow$ shift the transform."

EXAMPLES: $e^{at} \times$ POLYNOMIAL

Example 1: Find $\mathcal{L}\{e^{4t}t^2\}$

We know $\mathcal{L}\{t^2\} = 2/s^3$. Apply the rule with $a = 4$:

$$\mathcal{L}\{e^{4t}t^2\} = \left.\frac{2}{s^3}\right|_{s \to s-4} = \frac{2}{(s-4)^3}$$

Example 2: Find $\mathcal{L}\{e^{-2t}(1+3t)\}$

$$\mathcal{L}\{1+3t\} = \frac{1}{s} + \frac{3}{s^2}$$

$$\mathcal{L}\{e^{-2t}(1+3t)\} = \frac{1}{s+2} + \frac{3}{(s+2)^2}$$

Example 3: Find $\mathcal{L}\{e^{3t} \cdot 5\}$

$$\mathcal{L}\{5e^{3t}\} = \left.\frac{5}{s}\right|_{s \to s-3} = \frac{5}{s-3}$$

Examples: $e^{at} \times$ Trigonometric

Example 1: Find $\mathcal{L}\{e^{3t}\sin(2t)\}$

$\mathcal{L}\{\sin(2t)\} = 2/(s^2+4)$. Replace $s \to s-3$:

$$\mathcal{L}\{e^{3t}\sin(2t)\} = \frac{2}{(s-3)^2+4}$$

Example 2: Find $\mathcal{L}\{e^{-t}\cos(t)\}$

$\mathcal{L}\{\cos(t)\} = s/(s^2+1)$. Replace $s \to s+1$:

$$\mathcal{L}\{e^{-t}\cos(t)\} = \frac{s+1}{(s+1)^2+1}$$

Inverse direction (from the same table entries):

$$\mathcal{L}^{-1}\left\{\frac{4}{(s+2)^2+9}\right\} = \tfrac{4}{3}e^{-2t}\sin(3t)$$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} = e^{t}\cos(2t)$$

Practice: First Translation Rule

Problem 1 (Forward): Find $\mathcal{L}\{e^{2t}\sin(3t)\}$ and $\mathcal{L}\{t\,e^{-t}\}$.

$$\mathcal{L}\{e^{2t}\sin(3t)\} = \frac{3}{(s-2)^2+9}$$

$$\mathcal{L}\{t\,e^{-t}\} = \left.\frac{1}{s^2}\right|_{s\to s+1} = \frac{1}{(s+1)^2}$$

Problem 2 (Inverse):

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} = e^{-t}\sin(2t)$$

$$\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+9}\right\} = e^{3t}\cos(3t)$$

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Unit Step (Heaviside) Function

Definition

The **unit step function** (or Heaviside function) is:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

- $u(t-a)$ is "off" before $t = a$ and "on" for $t \geq a$
- Models a signal or forcing that **switches on at time** $t = a$
- $f(t) \cdot u(t-a)$ zeroes out $f(t)$ before $t = a$ and leaves it unchanged after

Examples:

- $u(t)$: switches on at $t = 0$ (standard Heaviside)
- $u(t-3)$: switches on at $t = 3$
- $u(t-1) - u(t-4)$: a pulse that is 1 for $1 \leq t < 4$, zero elsewhere

Piecewise Functions via Step Functions

Any piecewise-defined function can be written as a combination of step functions.

Example 1:

$$f(t) = \begin{cases} 0 & t < 3 \\ t - 3 & t \geq 3 \end{cases} \implies f(t) = (t-3)\,u(t-3)$$

Example 2:

$$g(t) = \begin{cases} 2 & 0 \leq t < 1 \\ t & t \geq 1 \end{cases} \implies g(t) = 2 + (t-2)\,u(t-1)$$

General pattern: switching from $f(t)$ to $h(t)$ at $t = a$:

$$\text{result}(t) = f(t) + [h(t) - f(t)]\,u(t-a)$$

Second Translation (t-Shifting) Rule

Theorem (Second Translation Rule)

If $\mathcal{L}\{f(t)\} = F(s)$, then for $a > 0$:

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$$

Proof sketch: Substitute $\tau = t - a$ in the integral:

$$\begin{aligned}\int_0^{\infty} f(t-a)\,u(t-a)\,e^{-st}\,dt &= \int_a^{\infty} f(t-a)\,e^{-st}\,dt \\ &= \int_0^{\infty} f(\tau)\,e^{-s(\tau+a)}\,d\tau = e^{-as}F(s)\end{aligned}$$

Inverse direction:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$$

The e^{-as} factor in s-domain $\Leftrightarrow$ a time delay of a units.

Examples: Laplace Transform of Step Forcing

Example 1: Find $\mathcal{L}\{(t-3)\,u(t-3)\}$
$f(t) = t$, $a = 3$, $\mathcal{L}\{t\} = 1/s^2$:

$$\mathcal{L}\{(t-3)\,u(t-3)\} = e^{-3s} \cdot \frac{1}{s^2}$$

Example 2: Find $\mathcal{L}\{\cos(t-\pi)\,u(t-\pi)\}$
$f(t) = \cos t$, $a = \pi$, $\mathcal{L}\{\cos t\} = s/(s^2+1)$:

$$\mathcal{L}\{\cos(t-\pi)\,u(t-\pi)\} = e^{-\pi s} \cdot \frac{s}{s^2+1}$$

Example 3: Find $\mathcal{L}\{e^{-(t-2)}\,u(t-2)\}$
$f(t) = e^{-t}$, $a = 2$, $\mathcal{L}\{e^{-t}\} = 1/(s+1)$:

$$\mathcal{L}\{e^{-(t-2)}\,u(t-2)\} = \frac{e^{-2s}}{s+1}$$

Inverse: t-Shifting

Strategy: Identify the e^{-as} factor, find $F(s)$, invert F to get $f(t)$, then shift.

Example 1: Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2+1}\right\}$

$F(s) = 1/(s^2+1)$, $a = 2$, so $f(t) = \sin t$:

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+1}\right\} = \sin(t-2)\,u(t-2)$$

Example 2: Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s+2}\right\}$

$F(s) = 1/(s+2)$, $a = 3$, so $f(t) = e^{-2t}$:

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+2}\right\} = e^{-2(t-3)}\,u(t-3)$$

Practice: Step Functions

Problem 1 (Forward): Find the Laplace transform of

$$f(t) = \begin{cases} 0 & t < 2 \\ \sin(t-2) & t \geq 2 \end{cases}$$

Solution: $f(t) = \sin(t-2)\, u(t-2)$, so:

$$\mathcal{L}\{f(t)\} = e^{-2s} \cdot \frac{1}{s^2+1}$$

Problem 2 (Inverse): Find $\mathcal{L}^{-1}\left\{ \dfrac{e^{-\pi s} \cdot s}{s^2+4} \right\}$

Solution: $F(s) = s/(s^2+4)$, $a = \pi$, so $f(t) = \cos(2t)$:

$$\mathcal{L}^{-1}\left\{ \frac{e^{-\pi s} \cdot s}{s^2+4} \right\} = \cos(2(t-\pi))\, u(t-\pi)$$

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Convolution: Definition

Definition

The **convolution** of two functions $f(t)$ and $g(t)$ is:

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Properties:

- **Commutative:** $f * g = g * f$
- **NOT pointwise:** $(f * g)(t) \neq f(t) \cdot g(t)$ in general

Quick Example: Compute $(1 * e^{-t})(t)$:

$$\begin{aligned} \int_0^t 1 \cdot e^{-(t-\tau)}\, d\tau &= e^{-t} \int_0^t e^{\tau}\, d\tau \\ &= e^{-t}[e^{\tau}]_0^t = e^{-t}(e^t - 1) = 1 - e^{-t} \end{aligned}$$

Convolution Theorem

Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then:

$$\mathcal{L}\{(f * g)(t)\} = F(s) \cdot G(s)$$

Important contrast:

- Convolution in time $\longleftrightarrow$ **multiplication** in s-domain
- Pointwise product in time $\not\longleftrightarrow$ multiplication in s-domain

Primary use — inverse transforms: When $Y(s) = F(s) \cdot G(s)$,

$$y(t) = \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = (f * g)(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau$$

Especially useful when partial fractions are impractical.

Using Convolution for Inverse LT

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$ using convolution.

Write $\dfrac{1}{s(s+1)} = \dfrac{1}{s} \cdot \dfrac{1}{s+1}$, so $f(t) = 1$ and $g(t) = e^{-t}$:

$$y(t) = \int_0^t 1 \cdot e^{-(t-\tau)}\, d\tau = e^{-t}[e^{\tau}]_0^t = 1 - e^{-t}$$

Verify via partial fractions: $\dfrac{1}{s(s+1)} = \dfrac{1}{s} - \dfrac{1}{s+1} \implies 1 - e^{-t}$ ✓

Both methods agree. Convolution is more general — it works even when partial fractions become complex.

Convolution Example 2

Find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}$ **using convolution.**

Write $F(s) = G(s) = 1/(s^2+1)$, so $f(t) = g(t) = \sin t$:

$$(f * g)(t) = \int_0^t \sin\tau \, \sin(t-\tau)\, d\tau$$

Use the identity $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$:

$$= \frac{1}{2}\int_0^t [\cos(2\tau - t) - \cos t]\, d\tau = \frac{1}{2}\left[\frac{\sin(2\tau - t)}{2} - \tau\cos t\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin t}{2} - t\cos t + \frac{\sin t}{2}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}(\sin t - t\cos t)$$

Practice: Convolution

Problem 1: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}$ using convolution.

Solution: $f(t) = 1; \quad g(t) = \dfrac{1}{2}\sin(2t)$

$$\begin{aligned} y(t) &= \int_0^t \tfrac{1}{2}\sin(2(t-\tau))\,d\tau \\ &= \tfrac{1}{2}\left[\tfrac{\cos(2(t-\tau))}{2}\right]_0^t = \tfrac{1-\cos(2t)}{4} \end{aligned}$$

Problem 2: Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$.

Solution: Write as $\dfrac{s}{s^2+1}\cdot\dfrac{1}{s^2+1}$, so $f = \cos t$, $g = \sin t$:

$$(f*g)(t) = \int_0^t \cos\tau\sin(t-\tau)\,d\tau = \frac{1}{2}t\sin t$$

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Motivation: Switched-On Forcing

Many physical systems are driven by forces that turn on (or off) at specific times:

- A switch closes and a voltage is applied at $t = t_0$
- A mechanical impulse applied over a short window
- Piecewise constant input in a control system

Such forcing functions are **discontinuous** — classical methods struggle. Laplace transforms handle them naturally via the Second Translation Rule.

Strategy:

1. Express the forcing using $u(t-a)$ notation
2. Apply $\mathcal{L}$ using $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$
3. Solve for $Y(s)$ algebraically
4. Invert using $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$

EXAMPLE: $y'' + y = \sin(t-\pi)\,u(t-\pi)$, $y(0) = 0,\ y'(0) = 0$

Step 1: Transform. Since $\mathcal{L}\{\sin(t-\pi)u(t-\pi)\} = e^{-\pi s}/(s^2 + 1)$:

$$(s^2 + 1)Y = \frac{e^{-\pi s}}{s^2 + 1}$$

Step 2: $Y(s) = \dfrac{e^{-\pi s}}{(s^2+1)^2} = e^{-\pi s} \cdot F(s)$ where $F(s) = \dfrac{1}{(s^2+1)^2}$.

Step 3: Identify $f(t)$.

From the convolution example:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \tfrac{1}{2}(\sin t - t\cos t)$$

Example: Final Answer and Verification

Step 4: Apply Second Translation Rule.

$$y(t) = f(t-\pi)\,u(t-\pi) = \frac{1}{2}\left[\sin(t-\pi) - (t-\pi)\cos(t-\pi)\right]\,u(t-\pi)$$

Simplify using $\sin(t-\pi) = -\sin t$ and $\cos(t-\pi) = -\cos t$:

Final Answer

$$y(t) = \frac{1}{2}\left[(t-\pi)\cos t - \sin t\right]\,u(t-\pi)$$

Verification:

- For $t < \pi$: $y(t) = 0$ — system is at rest before forcing begins ✓
- $y(0) = y'(0) = 0$ are built into the transform at Step 1 ✓

Practice Problems

Problem 1: Solve $y'' + y = u(t-\pi)$, $y(0) = 0, y'(0) = 0$.

Solution: $(s^2+1)Y = e^{-\pi s}/s \implies Y = e^{-\pi s} \cdot \dfrac{1}{s(s^2+1)}$

PF: $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$, $\quad f(t) = 1 - \cos t$

$$y(t) = [1 - \cos(t-\pi)]\, u(t-\pi) = [1 + \cos t]\, u(t-\pi)$$

Problem 2: Solve $y'' + 2y' + y = e^{-(t-1)}u(t-1)$, $y(0) = 0$, $y'(0) = 0$.

Solution: $(s+1)^2 Y = e^{-s}/(s+1) \implies Y = e^{-s}/(s+1)^3$
$\mathcal{L}^{-1}\{1/(s+1)^3\} = \frac{t^2}{2}e^{-t}$, so:

$$y(t) = \frac{(t-1)^2}{2}\, e^{-(t-1)}\, u(t-1)$$

Thank You