CALCULUS

Lecture 12: Laplace Transforms

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Why Laplace Transforms?

Solving differential equations directly can be tedious. Laplace transforms offer a systematic algebraic approach.

Key advantages:

- Converts a DE into an **algebraic equation —** easy to manipulate
- Initial conditions are **built in** from the start (no separate step)
- Handles **discontinuous** or piecewise forcing functions naturally

The transform pathway:

$$\underbrace{y'' + 3y' + 2y = e^{4t}}_{\text{ODE (hard)}} \xrightarrow{\mathcal{L}} \underbrace{\text{Algebraic equation in } Y(s)}_{\text{Easy to solve}} \xrightarrow{\mathcal{L}^{-1}} y(t)$$

Definition of the Laplace Transform

Definition

The **Laplace transform** of a function $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt = F(s)$$

provided the integral converges, which requires s to be sufficiently large.

- t: time variable (or independent variable); $f(t)$: function in **time domain**
- s: complex frequency parameter; $F(s)$: function in ***s*-domain**
- $f(t)$ and $F(s)$ form a **transform pair**: $f(t) \longleftrightarrow F(s)$

Example: $\mathcal{L}\{e^{at}\} = F(s)$ *exists for* $s > a$ *(we will derive this shortly).*

LINEARITY OF THE LAPLACE TRANSFORM

Linearity

For constants a, b and functions $f(t), g(t)$:

$$\mathcal{L}\{a\, f(t) + b\, g(t)\} = a\, F(s) + b\, G(s)$$

Proof: Follows directly from linearity of integration:

$$\int_0^\infty [af + bg]e^{-st}\, dt = a\int_0^\infty f\, e^{-st}\, dt + b\int_0^\infty g\, e^{-st}\, dt$$

Examples (using the table we will build):

$$\mathcal{L}\{3e^{2t} - 5\sin(4t)\} = \frac{3}{s-2} - \frac{20}{s^2+16}$$

$$\mathcal{L}\{2t^3 + 7\cos(3t)\} = \frac{12}{s^4} + \frac{7s}{s^2+9}$$

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$\mathcal{L}\{1\}$ from the Definition

Evaluate directly using the definition:

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \lim_{t\to\infty} \frac{e^{-st}}{-s} - \frac{e^0}{-s} = 0 + \frac{1}{s}$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0$$

(The limit $\lim_{t\to\infty} e^{-st} = 0$ holds because $s > 0$.)

$\mathcal{L}\{t\}$ AND $\mathcal{L}\{t^n\}$

Deriving $\mathcal{L}\{t\}$ by Integration by Parts: Let $u = t$, $dv = e^{-st}dt$:

$$\mathcal{L}\{t\} = \left[\frac{-t\,e^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = 0 + \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

General pattern (by repeated IBP or the Gamma function):

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad n = 0, 1, 2, \ldots, \quad s > 0$$

Examples:

$$\mathcal{L}\{t^2\} = \frac{2}{s^3} \qquad \mathcal{L}\{t^3\} = \frac{6}{s^4} \qquad \mathcal{L}\{t^5\} = \frac{120}{s^6}$$

$\mathcal{L}\{e^{at}\}$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} \cdot e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = 0 + \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad s > a$$

Examples:

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3} \qquad (s > 3)$$

$$\mathcal{L}\{e^{-5t}\} = \frac{1}{s+5} \qquad (s > -5)$$

$\mathcal{L}\{\sin(kt)\}$

Let $I = \mathcal{L}\{\sin(kt)\}$. Apply integration by parts twice (first: $u = \sin(kt)$, $dv = e^{-st}dt$):

$$I = \underbrace{\left[\frac{-e^{-st}\sin(kt)}{s}\right]_0^\infty}_{=0} + \frac{k}{s}\mathcal{L}\{\cos(kt)\}$$

Now apply integration by parts to $\mathcal{L}\{\cos(kt)\}$ ($u = \cos(kt)$, $dv = e^{-st}dt$):

$$\mathcal{L}\{\cos(kt)\} = \frac{1}{s} - \frac{k}{s}I$$

Substituting back:

$$I = \frac{k}{s}\left(\frac{1}{s} - \frac{k}{s}I\right) = \frac{k}{s^2} - \frac{k^2}{s^2}I \implies I\left(1 + \frac{k^2}{s^2}\right) = \frac{k}{s^2}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2 + k^2}$$

$\mathcal{L}\{\cos(kt)\}$

Use the derivative relation $\frac{d}{dt}[\sin(kt)] = k\cos(kt)$ and the derivative rule $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ (derived on the next slide):

$$\mathcal{L}\left\{\frac{d}{dt}[\sin(kt)]\right\} = s \cdot \mathcal{L}\{\sin(kt)\} - \sin(0)$$

$$k\,\mathcal{L}\{\cos(kt)\} = s \cdot \frac{k}{s^2+k^2} - 0$$

Dividing both sides by k:

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

Examples:

$$\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9} \qquad \mathcal{L}\{\cos(5t)\} = \frac{s}{s^2+25}$$

Standard Transform Table

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$

Using the Table: Examples

Apply the table with linearity to find transforms directly.

Example 1: Find $\mathcal{L}\{4t^2 - 3e^{-2t} + 5\sin(3t)\}$

$$= 4 \cdot \frac{2!}{s^3} - 3 \cdot \frac{1}{s+2} + 5 \cdot \frac{3}{s^2+9} = \frac{8}{s^3} - \frac{3}{s+2} + \frac{15}{s^2+9}$$

Example 2: Find $\mathcal{L}\{e^{-t}\cos(2t) + 3e^{5t}\}$

$$= \frac{s+1}{(s+1)^2+4} + \frac{3}{s-5}$$

Example 3: Find $\mathcal{L}\{6 - 2t^3 + 4\cos(t)\}$

$$= \frac{6}{s} - \frac{12}{s^4} + \frac{4s}{s^2+1}$$

Derivative Rule: $\mathcal{L}\{f'\}$ and $\mathcal{L}\{f''\}$

Derivation (integration by parts with $u = e^{-st}$, $dv = f'(t)dt$):

$$\mathcal{L}\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty + s \int_0^\infty f(t) e^{-st}\, dt$$
$$= (0 - f(0)) + sF(s)$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Applying the rule again to $f'' = (f')'$:

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0) = s[sF(s) - f(0)] - f'(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s\,f(0) - f'(0)$$

Derivative Rule: Verification

Check 1: Let $f(t) = e^{at}$, so $f'(t) = ae^{at}$ and $f(0) = 1$.

LHS: $\mathcal{L}\{ae^{at}\} = \dfrac{a}{s-a}$

RHS: $sF(s) - f(0) = \dfrac{s}{s-a} - 1 = \dfrac{s-(s-a)}{s-a} = \dfrac{a}{s-a} \quad \checkmark$

Check 2: Let $g(t) = \sin(kt)$, so $g'(t) = k\cos(kt)$ and $g(0) = 0$.

$$\text{LHS: } \mathcal{L}\{k\cos(kt)\} = \frac{ks}{s^2+k^2}$$

$$\text{RHS: } s \cdot \frac{k}{s^2+k^2} - 0 = \frac{ks}{s^2+k^2} \quad \checkmark$$

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Inverse LT — Definition and Table Lookup

Definition

Given $F(s)$, the **inverse Laplace transform** $\mathcal{L}^{-1}\{F(s)\} = f(t)$ is the unique function whose transform is $F(s)$.

Linearity: $\mathcal{L}^{-1}\{aF + bG\} = a\mathcal{L}^{-1}\{F\} + b\mathcal{L}^{-1}\{G\}$

Direct table lookup:

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3t \qquad (\text{since } \mathcal{L}\{t\} = 1/s^2)$$

$$\mathcal{L}^{-1}\left\{\frac{5}{s+4}\right\} = 5e^{-4t}$$

$$\mathcal{L}^{-1}\left\{\frac{6}{s^2+9}\right\} = 2\sin(3t) \qquad (\text{factor: } 6 = 2 \cdot 3,\ k = 3)$$

Completing the Square

When the denominator is a quadratic with no real roots, complete the square to match table entries.

Example 1: Find $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4s+8}\right\}$

$$s^2+4s+8=(s+2)^2+4$$

$$\frac{2}{(s+2)^2+4}=\frac{2}{(s+2)^2+2^2} \longrightarrow e^{-2t}\sin(2t)$$

Example 2: Find $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+2s+10}\right\}$

$$s^2+2s+10=(s+1)^2+9$$

$$\frac{s+1}{(s+1)^2+9} \longrightarrow e^{-t}\cos(3t)$$

Partial Fractions: Type 1 — Distinct Real Roots

For $F(s) = \frac{N(s)}{(s-a)(s-b)}$, write $\dfrac{N(s)}{(s-a)(s-b)} = \dfrac{A}{s-a} + \dfrac{B}{s-b}$.

Cover-up: multiply by $(s-a)$ and set $s = a$ to find A; similarly for B.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{6}{s(s+3)}\right\}$

$$\frac{6}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$A = \left.\frac{6}{s+3}\right|_{s=0} = 2, \qquad B = \left.\frac{6}{s}\right|_{s=-3} = -2$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2}{s+3}\right\} = 2 - 2e^{-3t}$$

Partial Fractions: Type 2 — Quadratic Denominator

For an irreducible quadratic factor, use $\dfrac{Bs+C}{s^2+ps+q}$ in the decomposition.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{s+3}{(s-2)(s^2+1)}\right\}$

$$\frac{s+3}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

Multiply through: $s+3 = A(s^2+1) + (Bs+C)(s-2)$

$$\begin{aligned} s=2: &\quad 5 = 5A \;\Rightarrow\; A = 1 \\ s=0: &\quad 3 = A - 2C = 1 - 2C \;\Rightarrow\; C = -1 \\ s^2 \text{ coeff}: &\quad 0 = A + B \;\Rightarrow\; B = -1 \end{aligned}$$

$$\frac{1}{s-2} - \frac{s+1}{s^2+1} = \frac{1}{s-2} - \frac{s}{s^2+1} - \frac{1}{s^2+1} \;\longrightarrow\; e^{2t} - \cos t - \sin t$$

Practice: Inverse Laplace Transform

Problem 1: Find $\mathcal{L}^{-1}\left\{\dfrac{6}{(s+1)(s+4)}\right\}$

Solution: $\dfrac{6}{(s+1)(s+4)} = \dfrac{A}{s+1} + \dfrac{B}{s+4}$

$$A = \frac{6}{s+4}\Big|_{s=-1} = 2, \quad B = \frac{6}{s+1}\Big|_{s=-4} = -2$$

$$f(t) = 2e^{-t} - 2e^{-4t}$$

Problem 2: Find $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+2s+5}\right\}$

Solution: $s^2 + 2s + 5 = (s+1)^2 + 4$, so $\dfrac{3}{(s+1)^2+4} = \dfrac{3}{2} \cdot \dfrac{2}{(s+1)^2+4}$

$$f(t) = \tfrac{3}{2}\, e^{-t} \sin(2t)$$

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The 4-Step Algorithm

Algorithm for solving IVPs via Laplace Transforms

1. **Transform** both sides of the ODE using $\mathcal{L}$; apply derivative rules $\mathcal{L}\{y'\} = sY - y(0)$, $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$
2. **Solve** the resulting algebraic equation for $Y(s)$
3. **Decompose** $Y(s)$ using partial fractions if needed
4. **Invert** to find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ using the table

The key insight: ICs are automatically encoded in Step 1 — no separate particular vs. homogeneous split needed.

EXAMPLE 1: $y' + 3y = 0, \quad y(0) = 2$

Step 1: Transform.

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 0 \implies [sY(s) - y(0)] + 3Y(s) = 0$$

Step 2: Substitute IC and solve for $Y(s)$.

$$sY - 2 + 3Y = 0$$

$$(s+3)Y = 2 \implies Y(s) = \frac{2}{s+3}$$

Step 4: Invert.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} = 2e^{-3t}$$

Quick check: $y' + 3y = -6e^{-3t} + 6e^{-3t} = 0$ ✓ *and* $y(0) = 2$ ✓

Example 2: $y' + 3y = 6, \quad y(0) = 0$ — Setup

Step 1: Transform.

$$[sY - 0] + 3Y = \mathcal{L}\{6\} = \frac{6}{s}$$

Step 2: Solve for $Y(s)$.

$$(s+3)Y = \frac{6}{s} \implies Y(s) = \frac{6}{s(s+3)}$$

Step 3: Partial fractions.

$$\frac{6}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$A = \left.\frac{6}{s+3}\right|_{s=0} = 2 \qquad B = \left.\frac{6}{s}\right|_{s=-3} = -2$$

$$Y(s) = \frac{2}{s} - \frac{2}{s+3}$$

Example 2: Inverse Transform and Solution

Step 4: Invert.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} = 2 - 2e^{-3t}$$

Verification:

$$\begin{aligned} y'(t) &= 6e^{-3t} \\ y' + 3y &= 6e^{-3t} + 3(2 - 2e^{-3t}) = 6e^{-3t} + 6 - 6e^{-3t} = 6 \quad \checkmark \\ y(0) &= 2 - 2 = 0 \quad \checkmark \end{aligned}$$

Note: As $t \to \infty$, $y \to 2$ — the steady state equals the constant forcing divided by the coefficient: $6/3 = 2$.

EXAMPLE 3: $y' + 3y = 13\sin(2t), \quad y(0) = 6$ — SETUP

Step 1: $[sY - 6] + 3Y = \dfrac{26}{s^2+4}$ **Step 2:** $(s+3)Y = \dfrac{26}{s^2+4} + 6$

$$\implies Y(s) = \frac{26}{(s+3)(s^2+4)} + \frac{6}{s+3}$$

Step 3: Partial fractions for the first term.

$$\frac{26}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

Multiply through: $26 = A(s^2+4) + (Bs+C)(s+3)$

$$s = -3 : 26 = 13A \implies A = 2$$
$$s = 0 : 26 = 4A + 3C = 8 + 3C \implies C = 6$$
$$s^2 : 0 = A + B \implies B = -2$$

Example 3: Final Solution

Combining the partial fractions:

$$\frac{26}{(s+3)(s^2+4)} = \frac{2}{s+3} + \frac{-2s+6}{s^2+4} = \frac{2}{s+3} - \frac{2s}{s^2+4} + \frac{6}{s^2+4}$$

Adding back $\frac{6}{s+3}$:

$$Y(s) = \frac{8}{s+3} - \frac{2s}{s^2+4} + \frac{3 \cdot 2}{s^2+4}$$

Step 4: Invert.

$$\boxed{y(t) = 8e^{-3t} - 2\cos(2t) + 3\sin(2t)}$$

Check at $t = 0$: $y(0) = 8 - 2 + 0 = 6$ ✓

Second-Order Derivative Rule

For a second-order ODE, we need $\mathcal{L}\{y''\}$.

Second-Order Transform Rule

$$\mathcal{L}\{y''(t)\} = s^2 Y(s) - s\,y(0) - y'(0)$$

Derivation: Apply $\mathcal{L}\{f'\} = sF - f(0)$ *twice, treating* $y'' = (y')'$*:*

$$\mathcal{L}\{y''\} = s\,\mathcal{L}\{y'\} - y'(0) = s[sY - y(0)] - y'(0)$$

Both initial conditions $y(0)$ and $y'(0)$ appear automatically.

The ODE $ay'' + by' + cy = g(t)$ transforms to:

$$a[s^2 Y - sy(0) - y'(0)] + b[sY - y(0)] + cY = G(s)$$

EXAMPLE 4: $y'' + 3y' + 2y = e^{4t}$, $y(0) = y'(0) = 0$ — SETUP

Step 1: Transform.

$$[s^2Y - 0 - 0] + 3[sY - 0] + 2Y = \frac{1}{s-4}$$

$$(s^2 + 3s + 2)Y = \frac{1}{s-4}$$

Step 2: Solve for $Y(s)$.

$$Y(s) = \frac{1}{(s^2 + 3s + 2)(s-4)} = \frac{1}{(s+1)(s+2)(s-4)}$$

Step 3: Set up partial fractions.

$$\frac{1}{(s+1)(s+2)(s-4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-4}$$

Example 4: Partial Fractions

Multiply through: $1 = A(s+2)(s-4) + B(s+1)(s-4) + C(s+1)(s+2)$

Cover-up at each root:

$$\begin{aligned} s=-1: &\quad 1 = A(1)(-5) = -5A \implies A = -\tfrac{1}{5} \\ s=-2: &\quad 1 = B(-1)(-6) = 6B \implies B = \tfrac{1}{6} \\ s=4: &\quad 1 = C(5)(6) = 30C \implies C = \tfrac{1}{30} \end{aligned}$$

$$Y(s) = \frac{-1/5}{s+1} + \frac{1/6}{s+2} + \frac{1/30}{s-4}$$

Example 4: Final Solution

Step 4: Invert.

$$\boxed{y(t) = -\frac{1}{5}e^{-t} + \frac{1}{6}e^{-2t} + \frac{1}{30}e^{4t}}$$

Verification at $t = 0$:

$$y(0) = -\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{30} = \frac{-6+5+1}{30} = 0 \quad \checkmark$$

$$y'(t) = \tfrac{1}{5}e^{-t} - \tfrac{1}{3}e^{-2t} + \tfrac{2}{15}e^{4t}$$

$$y'(0) = \tfrac{1}{5} - \tfrac{1}{3} + \tfrac{2}{15} = \frac{3-5+2}{15} = 0 \quad \checkmark$$

Practice Problems

Problem 1: Solve $y' + 2y = 4, \quad y(0) = 1.$

Solution: $(s+2)Y - 1 = 4/s \implies Y = (s+4)/(s(s+2))$
PF: $A = 4/(s+2)|_{s=0} = 2, \; B = (s+4)/s|_{s=-2} = -1$

$$y(t) = 2 - e^{-2t} \qquad \text{[Check: } y(0) = 1 \checkmark]$$

Problem 2: Solve $y'' - y = \cos t, \quad y(0) = 0, y'(0) = 0.$

Solution: $(s^2-1)Y = s/(s^2+1) \implies Y = s/((s-1)(s+1)(s^2+1))$
PF: $A = 1/4$, $B = 1/4$, $C = -1/2$, $D = 0$

$$y(t) = \frac{1}{4}e^{t} + \frac{1}{4}e^{-t} - \frac{1}{2}\cos t \qquad \text{[Check: } y(0) = 0 \checkmark]$$

Thank You