CALCULUS

Lecture 11

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Linear First-Order DEs

Definition

A differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is called a **linear first-order DE** in y.

- If $g(x) = 0$: **homogeneous**. If $g(x) \neq 0$: **nonhomogeneous**.
- e.g. $y' + 2y = e^x$ is linear; $y' + y^2 = x$ is **not** (y^2 makes it nonlinear).

Standard Form

Divide by the leading coefficient $a_1(x)$:

$$\frac{dy}{dx} + P(x)\,y = f(x)$$

where $P(x) = \dfrac{a_0(x)}{a_1(x)}$ and $f(x) = \dfrac{g(x)}{a_1(x)}$.

- We assume $P(x)$ and $f(x)$ are continuous on an interval I.
- Points where $a_1(x) = 0$ are called **singular points**.
- **Example:** For $xy' + y = x$, dividing by x gives $y' + \frac{1}{x}y = 1$. The point $x = 0$ is a singular point — the solution is valid on $(0, \infty)$ or $(-\infty, 0)$ only.

Structure of Solutions

Linearity lets us split the problem into two simpler subproblems — homogeneous + particular:

$$y = y_c + y_p$$

- y_c: solution of the **homogeneous** equation $\dfrac{dy}{dx} + P(x)y = 0$
- y_p: any **particular solution** of $\dfrac{dy}{dx} + P(x)y = f(x)$

The homogeneous solution y_c carries the arbitrary constant; y_p accounts for the forcing term $f(x)$.

Homogeneous Solution y_c

Solve

$$\frac{dy}{dx} + P(x)\,y = 0$$

Separate variables and integrate:

$$\frac{dy}{y} = -P(x)\,dx \quad \Rightarrow \ln|y| = -\int P(x)\,dx + C \Rightarrow \boxed{y_c = C\,e^{-\int P(x)\,dx}}$$

Integrating Factor

To solve $\dfrac{dy}{dx} + P(x)\,y = f(x)$, define the **integrating factor**

$$\mu(x) = e^{\int P(x)\,dx}$$

Note: $\mu(x) = e^{\int P\,dx} = \dfrac{1}{y_c}$ — it is exactly the reciprocal of the homogeneous solution. Multiplying by μ "unwraps" y_c.

Multiply the equation by $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)\,y = \mu(x)f(x)$$

KEY OBSERVATION

Check that $\frac{d}{dx}[\mu y] = \mu \frac{dy}{dx} + \mu' y$ and since $\mu' = \mu P$:

$$\frac{d}{dx}[\mu y] = \mu \frac{dy}{dx} + \mu P\, y = \mu \frac{dy}{dx} + \mu(x) P(x)\, y$$

So the equation becomes:

$$\frac{d}{dx}\big[\mu(x)\, y\big] = \mu(x)\, f(x)$$

Integrate both sides:

$$\mu(x)\, y = \int \mu(x)\, f(x)\, dx + C$$

General Solution

Solution via Integrating Factor

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)\, f(x)\, dx + C\right), \qquad \mu(x) = e^{\int P(x)\, dx}$$

Solving Procedure

Steps to solve $y' + P(x)y = f(x)$:

1. Write in standard form: identify $P(x)$ and $f(x)$.
2. Compute $\mu(x) = e^{\int P(x)\,dx}$.
3. Multiply through by μ; the left side becomes $\dfrac{d}{dx}[\mu y]$.
4. Integrate both sides and solve for y.

Example 1 — Constant Forcing

Solve

$$\frac{dy}{dx} - 3y = 6$$

Standard form: $P(x) = -3$, $f(x) = 6$

Integrating factor:

$$\mu(x) = e^{\int -3\,dx} = e^{-3x}$$

Multiply through (left side collapses to a product rule):

$$\frac{d}{dx}\left(e^{-3x}y\right) = 6e^{-3x}$$

Example 1 — Constant Forcing (Solution)

Integrate:

$$e^{-3x}y = \int 6e^{-3x}\,dx = -2e^{-3x} + C$$

Divide by e^{-3x}:

$$\boxed{y = -2 + Ce^{3x}}$$

- $y = -2$ is the particular solution (y_p).
- Ce^{3x} is the homogeneous solution (y_c).

Example 2 — Variable Coefficient

Solve

$$x\frac{dy}{dx} - 4y = x^6 e^x$$

Divide by x to get standard form:

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x$$

$P(x) = -\frac{4}{x}$

Integrating factor:

$$\mu(x) = e^{\int -4/x\, dx} = e^{-4\ln x} = x^{-4}$$

Example 2 — Variable Coefficient (Solution)

Multiply through by x^{-4}:

$$\frac{d}{dx}\left(x^{-4}y\right) = x^5e^x \cdot x^{-4} = xe^x$$

Integrate (by parts: $\int xe^x\,dx = xe^x - e^x$):

$$x^{-4}y = xe^x - e^x + C$$

Multiply by x^4:

$$\boxed{y = x^5e^x - x^4e^x + Cx^4}$$

IVP Example

Solve

$$\frac{dy}{dx} + y = x, \qquad y(0) = 4$$

$P(x) = 1$, so $\mu(x) = e^x$

$$\frac{d}{dx}(e^x y) = xe^x$$

Integrate (by parts: $u = x$, $dv = e^x\, dx$):

$$e^x y = xe^x - e^x + C \quad \Rightarrow \quad y = x - 1 + Ce^{-x}$$

IVP Example — Applying Initial Condition

General solution: $y = x - 1 + Ce^{-x}$
Apply $y(0) = 4$:

$$4 = 0 - 1 + C \cdot 1 \quad \Rightarrow \quad C = 5$$

$$\boxed{y = x - 1 + 5e^{-x}}$$

Transient Terms

From the solution $y = x - 1 + 5e^{-x}$:

- $x - 1$ is the **particular solution** — it describes the long-term behaviour.
- $5e^{-x}$ is the **homogeneous solution**, which vanishes as $x \to \infty$.

Transient Term

A term in the solution that $\to 0$ as $x \to \infty$ is called a **transient term**.

Transient terms carry the effect of initial conditions, but eventually die out. The particular solution dominates long-term behaviour.

Practice Problem 1

Solve $\dfrac{dy}{dx} - y = e^{3x}$.
$P(x) = -1$, $\mu(x) = e^{-x}$, so:

$$\frac{d}{dx}(e^{-x}y) = e^{2x}$$

Integrate:

$$e^{-x}y = \frac{e^{2x}}{2} + C$$

$$\boxed{y = \frac{1}{2}e^{3x} + Ce^{x}}$$

No transient term (both terms grow as $x \to \infty$).

Practice Problem 2

Solve

$$\frac{dy}{dx} + 3y = x$$

$P(x) = 3$, $\mu(x) = e^{3x}$:

$$\frac{d}{dx}(e^{3x}y) = xe^{3x}$$

Integrate by parts ($u = x$, $dv = e^{3x}dx$):

$$e^{3x}y = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$$

$$\boxed{y = \frac{x}{3} - \frac{1}{9} + Ce^{-3x}}$$

Transient term: $Ce^{-3x} \to 0$ as $x \to \infty$.

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Higher-Order Linear DEs

Consider the **homogeneous** linear equation with **constant** coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

- $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$

Key idea: Try a solution of the form

$$y = e^{mx}$$

Then $y^{(k)} = m^k e^{mx}$. Substituting and factoring out $e^{mx} \neq 0$ reduces the problem to a **polynomial equation** in m.

Auxiliary Equation

For the second-order case

$$ay'' + by' + cy = 0$$

substitute $y = e^{mx}$:

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

Factor out e^{mx}:

$$e^{mx}(am^2 + bm + c) = 0$$

Since $e^{mx} \neq 0$:

Auxiliary (Characteristic) Equation

$$am^2 + bm + c = 0$$

Solve for m; each root gives a solution e^{mx}.

Case I — Distinct Real Roots

If the auxiliary equation has two **distinct real roots** $m_1 \neq m_2$:

$$y_1 = e^{m_1 x}, \qquad y_2 = e^{m_2 x}$$

These are linearly independent, so the general solution is:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case II — Repeated Root

If the auxiliary equation has a **repeated root** $m_1 = m_2 = m$:

$$y_1 = e^{mx}$$

Finding y_2: Try $y_2 = u(x)e^{mx}$. Substituting into $ay'' + by' + cy = 0$ and using $am^2 + bm + c = 0$ reduces to $u'' = 0$, so $u = x$ (taking simplest non-trivial solution):

$$y_2 = xe^{mx}$$

General solution:

$$y = (c_1 + c_2 x)e^{mx}$$

Case III — Complex Roots

If the auxiliary equation has **complex roots** $m = a \pm ib$ (with $b \neq 0$):
Using Euler's formula:

$$e^{(a \pm ib)x} = e^{ax}(\cos bx \pm i \sin bx)$$

Taking real and imaginary parts gives two real solutions:

$$y_1 = e^{ax} \cos bx, \qquad y_2 = e^{ax} \sin bx$$

General solution:

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

Example 1 — Distinct Real Roots

Solve

$$2y'' - 5y' - 3y = 0$$

Auxiliary equation:

$$2m^2 - 5m - 3 = 0$$

Factor: $(2m + 1)(m - 3) = 0$

Roots: $m_1 = -\dfrac{1}{2}, \quad m_2 = 3$

$$\boxed{y = c_1 e^{-x/2} + c_2 e^{3x}}$$

Example 2 — Repeated Root

Solve

$$y'' - 10y' + 25y = 0$$

Auxiliary equation: $m^2 - 10m + 25 = (m-5)^2 = 0$ Repeated root: $m = 5$

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

Verify $y_2 = xe^{5x}$:

$$y_2' = e^{5x} + 5xe^{5x}, \qquad y_2'' = 10e^{5x} + 25xe^{5x}$$

$$y_2'' - 10y_2' + 25y_2 = (10 + 25x) - 10(1 + 5x) + 25x = 0 \checkmark$$

Example 3 — Complex Roots

Solve

$$y'' + 4y' + 7y = 0$$

Auxiliary equation:

$$m^2 + 4m + 7 = 0$$

Quadratic formula: $m = \dfrac{-4 \pm \sqrt{16 - 28}}{2} = -2 \pm \sqrt{3}\, i$

So $a = -2$, $b = \sqrt{3}$

$$\boxed{y = e^{-2x}\left(c_1 \cos \sqrt{3}\, x + c_2 \sin \sqrt{3}\, x\right)}$$

IVP Example

Solve

$$4y'' + 4y' + 17y = 0, \qquad y(0) = -1, \quad y'(0) = 2$$

Auxiliary equation:

$$4m^2 + 4m + 17 = 0$$

Roots: $m = \dfrac{-4 \pm \sqrt{16 - 272}}{8} = -\dfrac{1}{2} \pm 2i$

General solution:

$$y = e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x)$$

IVP Example — Applying Initial Conditions

From $y(0) = -1$: $\quad c_1 = -1$

Differentiate using the product rule:

$$y' = -\tfrac{1}{2}e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x) + e^{-x/2}(-2c_1 \sin 2x + 2c_2 \cos 2x)$$

At $x = 0$: $\quad y'(0) = -\tfrac{1}{2}c_1 + 2c_2 = 2$

$$\tfrac{1}{2} + 2c_2 = 2 \quad \Rightarrow \quad c_2 = \frac{3}{4}$$

$$\boxed{y = e^{-x/2}\left(-\cos 2x + \frac{3}{4}\sin 2x\right)}$$

Two Important Special Equations

$y'' + k^2 y = 0$

Aux: $m = \pm ki$

$$y = c_1 \cos kx + c_2 \sin kx$$

Pure oscillation.

$y'' - k^2 y = 0$

Aux: $m = \pm k$

$$y = c_1 e^{kx} + c_2 e^{-kx}$$

Exponential growth/decay.

Higher-Order Equations

For the nth-order homogeneous equation, the auxiliary polynomial has degree n.

- Each **distinct real root** m contributes e^{mx}.
- A root m of **multiplicity** k contributes:
  $e^{mx}, \; xe^{mx}, \; \ldots, \; x^{k-1}e^{mx}$
- Each pair of **complex roots** $a \pm ib$ contributes $e^{ax} \cos bx$ and $e^{ax} \sin bx$.

The general solution is a linear combination of n linearly independent solutions.

HIGHER-ORDER EXAMPLE

Solve $y''' - 3y'' + 3y' - y = 0$.
Auxiliary equation:

$$m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0$$

Triple root $m = 1$ (multiplicity 3) contributes e^x, xe^x, x^2e^x:

$$\boxed{y = (c_1 + c_2x + c_3x^2)e^x}$$

Practice Problem 1

Solve

$$y'' + 4y = 0$$

Auxiliary equation:

$$m^2 + 4 = 0 \quad \Rightarrow \quad m = \pm 2i$$

Complex roots: $a = 0$, $b = 2$

$$\boxed{y = c_1 \cos 2x + c_2 \sin 2x}$$

Pure oscillation — no damping.

Practice Problem 2

Solve

$$y'' + 2y' + 5y = 0$$

Auxiliary equation:

$$m^2 + 2m + 5 = 0 \quad \Rightarrow \quad m = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$$

$a = -1$, $b = 2$

$$\boxed{y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)}$$

Damped oscillation — amplitude decays as $x \to \infty$.

Practice Problem 3

Solve

$$y'' - 2y' + 2y = 0$$

Auxiliary equation:

$$m^2 - 2m + 2 = 0 \quad \Rightarrow \quad m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$a = 1$, $b = 1$

$$\boxed{y = e^x(c_1 \cos x + c_2 \sin x)}$$

Growing oscillation — amplitude increases as $x \to \infty$.

Practice Problem 4

Solve

$$y'' - 6y' + 9y = 0$$

Auxiliary equation:

$$m^2 - 6m + 9 = (m-3)^2 = 0$$

Repeated root: $m = 3$

$$\boxed{y = (c_1 + c_2 x)e^{3x}}$$

Thank you :)