CALCULUS

Lecture 10

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Why Differential Equations?

- Differential equations arise when we try to describe physical phenomena mathematically.
- Many scientific and engineering models involve relationships between variables and their rates of change.
- The derivative $\frac{dy}{dx}$ of a function $y = f(x)$ is itself another function.

From a Function to a DE

- Example:

$$y = e^{0.1x^2} \quad \Rightarrow \quad \frac{dy}{dx} = 0.2xe^{0.1x^2}$$

- Replacing $e^{0.1x^2}$ by y gives

$$\frac{dy}{dx} = 0.2xy$$

- Key question: **What function $y = f(x)$ satisfies this equation?**

Main Problem of Differential Equations

- In differential equations we are given an equation involving derivatives.
- The objective is to determine the unknown function.
- This is similar to the reverse problem in calculus:
  - Given a derivative, find the antiderivative.
- Before solving differential equations, we must understand the basic terminology and classification.

Definition: Differential Equation

Differential Equation

An equation containing the derivatives of one or more dependent variables with respect to one or more independent variables is called a **differential equation**.

- Dependent variable: the unknown function.
- Independent variable: the variable with respect to which derivatives are taken.
- Differential equations appear in many scientific models.
- **Example:** $\dfrac{dy}{dx} + 3y = x$ — dependent variable: y, independent variable: x.

Ordinary vs Partial Differential Equations

Ordinary Differential Equation (ODE)

A differential equation involving derivatives with respect to a **single independent variable**.

Example: $\dfrac{dy}{dx} + 6y = e^{-x}$ involves one variable x.

Partial Differential Equation (PDE)

A differential equation involving **partial derivatives** of functions with respect to **two or more independent variables**.

Example: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ involves two variables x and y.

Examples of ODEs

- Examples of ordinary differential equations:

$$\frac{dy}{dx} + 6y = e^{-x}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 3x + 2y$$

Examples of PDEs

- Examples of partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- Multiple dependent and independent variables may appear.

Derivative Notations

Leibniz / Prime

$$\frac{dy}{dx}, \quad y', \quad y^{(n)} = \frac{d^n y}{dx^n}$$

- Most common in mathematics and engineering.

Dot / Subscript

$$\dot{s}, \quad \ddot{s} = -32$$

$$u_{xx} + u_{yy} = 0$$

- Dot: derivatives w.r.t. time.
- Subscript: partial derivatives.

Order of a Differential Equation

Order

The **order** of a differential equation is the order of the highest derivative present.

- **ODE Example:**

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 4y = e^x$$

Highest derivative: $\frac{d^2y}{dx^2}$ $\Rightarrow$ **Order = 2**

- **PDE Example:**

$$2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

Highest derivative order: 4 $\Rightarrow$ **Order = 4**

Differential Form

A first-order differential equation can be written in differential form:

$$M(x, y)dx + N(x, y)dy = 0$$

- Example:
$$(y^2 - x)dx + 4xdy = 0$$
- Dividing by dx gives
$$y^2 - x + 4x\frac{dy}{dx} = 0$$

General and Normal Form of an ODE

General Form

An nth-order ODE can be written as

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

Normal Form

Solving for the highest derivative gives the **normal form**:

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

- First-order: $\dfrac{dy}{dx} = f(x, y)$
- Second-order: $\dfrac{d^2 y}{dx^2} = f(x, y, y')$

Linear Differential Equations

An nth-order ODE is **linear** if it can be written as

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

- The dependent variable and its derivatives appear only to the **first power**.
- Coefficients depend only on the independent variable.

Examples of Linear ODEs

- Linear first-order equation:

$$\frac{dy}{dx} + 3y = x^2$$

- Linear second-order equation:

$$y'' - 2y' + y = 0$$

- Linear third-order equation:

$$x^3 \frac{d^3y}{dx^3} + 3x\frac{dy}{dx} + 5y = e^x$$

Nonlinear Differential Equations

A differential equation is **nonlinear** if any of the following appear:

- Powers such as $(y')^2$
- Coefficients depending on y
- Nonlinear functions such as $\sin y$

Examples:

$$(1-y)y' = 2y - e^x \qquad \text{(coefficient depends on } y\text{)}$$

$$\frac{d^2y}{dx^2} + \sin y = 0 \qquad \text{(nonlinear function of } y\text{)}$$

Homogeneous and Nonhomogeneous DEs

For a linear ODE

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Homogeneous

If $g(x) = 0$, the equation is called **homogeneous**:

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$

Nonhomogeneous

If $g(x) \neq 0$, the equation is **nonhomogeneous**.

Examples: Homogeneous vs Nonhomogeneous

Classify each equation:

- $y'' - 2y' + y = 0$ homogeneous ($g(x) = 0$)
- $y'' - 2y' + y = e^x$ nonhomogeneous ($g(x) = e^x$)
- $\dfrac{dy}{dx} + 3y = 0$ homogeneous ($g(x) = 0$)
- $\dfrac{dy}{dx} + 3y = x^2$ nonhomogeneous ($g(x) = x^2$)

DEFINITION: SOLUTION OF AN ODE

Solution

A function $f(x)$ with at least n continuous derivatives that satisfies the differential equation is called a **solution**.

$$F(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0$$

- The solution must satisfy the equation for all x in an interval.

Verification of a Solution

Verify that $y = \dfrac{x^4}{16}$ satisfies $\dfrac{dy}{dx} = xy^{1/2}$.

Compute LHS:

$$\frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$$

Compute RHS:

$$x\left(\frac{x^4}{16}\right)^{1/2} = x \cdot \frac{x^2}{4} = \frac{x^3}{4}$$

Compare: LHS $=$ RHS $\checkmark$ Hence $y = \dfrac{x^4}{16}$ is a solution.

Solution Curve

- The graph of a solution of a differential equation is called a **solution curve**.
- A solution must be differentiable over the interval of definition.
- Example: $y = \frac{1}{x}$
- Domain of the function: $x \neq 0$
- As a solution of a differential equation, it is valid only on intervals not containing 0.

Explicit Solutions

Explicit Solution

A solution where the dependent variable is expressed directly as a function of the independent variable.

Examples:

$$y = \frac{x^4}{16}$$

$$y = xe^x$$

$$y = \frac{1}{x}$$

Implicit Solutions

Implicit Solution

A relation of the form

$$G(x, y) = 0$$

that defines a solution of a differential equation.

Example:

$$x^2 + y^2 = 25$$

Families and Particular Solutions

- Solutions often contain arbitrary constants, producing a **family of solutions**.

Example: $y = cx - x\cos x$ (each value of c gives a different curve)

Particular Solution

A solution obtained by assigning a specific value to the arbitrary constant.

If $c = 0$:

$$y = -x\cos x$$

Singular Solutions

- Some differential equations have solutions that are not part of the general family.
- These are called **singular solutions**.

Example:

$$\frac{dy}{dx} = xy^{1/2}$$

Family of solutions:

$$y = \left(\frac{x^2}{4} + c\right)^2$$

But $y = 0$ is also a solution that cannot be obtained from the family.

System of Differential Equations

System of ODEs

Two or more differential equations involving several unknown functions.

Example:

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y)$$

- The solution consists of functions $x(t)$ and $y(t)$ satisfying both equations.

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MOTIVATION

- In many applications we want a solution of a differential equation that also satisfies additional conditions.
- These additional conditions specify the value of the unknown function or its derivatives at a particular point.
- Such conditions are called **side conditions**.
- A common problem of this type is called an **Initial-Value Problem (IVP)**.

Definition: Initial-Value Problem

Initial-Value Problem

On an interval I containing x_0, the problem

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

subject to

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

is called an *n***th-order initial-value problem**.

- The specified values are called **initial conditions**.

Interpretation of Initial Conditions

- Initial conditions specify the value of the function and its derivatives at a point x_0.
- Example initial conditions:

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$y''(x_0) = y_2$$

- These conditions determine a specific solution from a family of solutions.

First-Order IVP

A first-order IVP has the form

$$\frac{dy}{dx} = f(x, y)$$

subject to

$$y(x_0) = y_0$$

- Geometrically, we seek a solution curve passing through the point (x_0, y_0).

Second-Order IVP

A second-order IVP has the form

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

subject to

$$y(x_0) = y_0, \qquad y'(x_0) = y_1$$

- The solution curve must pass through (x_0, y_0).
- The slope of the curve at that point must be y_1.

Example: First-Order IVP

The differential equation

$$y' = y$$

has a one-parameter family of solutions

$$y = ce^x$$

Suppose the initial condition is

$$y(0) = 3$$

Substituting:

$$3 = ce^0 = c$$

Therefore the solution is

$$y = 3e^x$$

Example: Another Initial Condition

Consider the same differential equation

$$y' = y$$

with initial condition

$$y(1) = -2$$

Substituting:

$$-2 = ce$$

$$c = -2e^{-1}$$

Hence the solution is

$$y = -2e^{x-1}$$

Interval of Definition

Consider the differential equation

$$y' + 2xy^2 = 0$$

One family of solutions is

$$y = \frac{1}{x^2 + c}$$

If the initial condition is

$$y(0) = -1$$

then

$$c = -1$$

Thus the solution becomes

$$y = \frac{1}{\quad}$$

Domain vs Interval of Definition

Three different concepts appear:

- **Domain of the function** All values of x for which the function is defined.
- **Interval of definition of a solution** Any interval where the function satisfies the differential equation.
- **Interval of the IVP solution** Must contain the initial point.

Example:

$$y = \frac{1}{x^2 - 1}$$

Largest interval containing $x = 0$:

$$(-1, 1)$$

Example: Second-Order IVP

Given differential equation

$$x'' + 16x = 0$$

General solution:

$$x = c_1 \cos(4t) + c_2 \sin(4t)$$

Initial conditions:

$$x(\pi/2) = -2$$

$$x'(\pi/2) = 1$$

Solving the IVP

From $x(\pi/2) = -2$:

$$c_1 \cos(2\pi) + c_2 \sin(2\pi) = c_1 = -2$$

Now differentiate the general solution:

$$x'(t) = -4c_1 \sin(4t) + 4c_2 \cos(4t)$$

Apply $x'(\pi/2) = 1$:

$$-4c_1 \sin(2\pi) + 4c_2 \cos(2\pi) = 4c_2 = 1 \quad \Rightarrow \quad c_2 = \frac{1}{4}$$

Final solution:

$$\boxed{x(t) = -2\cos(4t) + \tfrac{1}{4}\sin(4t)}$$

Two Important Questions

When studying an initial-value problem, two questions arise:

- **Existence**: Does a solution exist?
- **Uniqueness**: Is the solution unique?

In other words:

- Does a solution curve pass through (x_0, y_0)?
- Is there exactly one such curve?

Example: Multiple Solutions

Consider the IVP

$$\frac{dy}{dx} = xy^{1/2}, \qquad y(0) = 0$$

Two solutions are:

$$y = 0$$

$$y = \frac{x^4}{16}$$

- Both curves pass through $(0, 0)$.
- Hence the IVP has more than one solution.

Existence and Uniqueness Theorem

Theorem

Let R be a rectangular region containing the point (x_0, y_0).
If

- $f(x, y)$ is continuous in R
- $\frac{\partial f}{\partial y}$ is continuous in R

then there exists an interval I_0 around x_0 in which the IVP

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

has a **unique solution**.

Interpretation of the Theorem

- The theorem guarantees existence and uniqueness **locally** near (x_0, y_0).
- The theorem does not specify the size of the interval of validity.
- **Connection to the counter-example:** For $\frac{dy}{dx} = xy^{1/2}$,

$$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y}}$$

  is **discontinuous** at $y = 0$ — the theorem does not apply at $(0, 0)$, which is precisely why two solutions exist.

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What is a Mathematical Model?

- A **mathematical model** is a mathematical description of a real-world system or phenomenon.
- The description can be a function, equation, or a system of equations.
- Often the model describes how quantities change over time.
- Many mathematical models involve **differential equations** because they express rates of change.

Example Idea

Leonardo da Vinci studied falling bodies by observing water drops falling at equal time intervals and deduced the relation

$$v = gt$$

for velocity of a falling object.

Purpose of Mathematical Models

Mathematical models are used to:

- Understand behavior of physical systems.
- Predict future outcomes.
- Analyze relationships between variables.
- Test hypotheses using mathematical analysis.

Systems Modelled by DEs

Differential equations naturally describe:

- Population growth
- Radioactive decay
- Spread of disease
- Chemical reactions
- Temperature change

Steps in Building a Mathematical Model

1. Identify the important variables of the system.
2. Make reasonable assumptions about the system.
3. Translate assumptions into mathematical relationships.
4. Formulate differential equations.
5. Solve the equations.
6. Compare predictions with experimental data.

Example: Population Growth Model

Assumption: The rate of growth is proportional to the current population.

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

Solution: Separate and integrate:

$$\int \frac{dP}{P} = \int k\,dt \quad \Rightarrow \quad P(t) = P_0 e^{kt}$$

- $k > 0$: exponential growth $\quad$ $k < 0$: exponential decay
- This same model describes radioactive decay and cooling.

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A Simple Case

Consider the differential equation

$$\frac{dy}{dx} = g(x)$$

- If $g(x)$ is continuous, the solution is obtained by direct integration:
$$y = \int g(x)\, dx = G(x) + c$$
- Example: $\quad \dfrac{dy}{dx} = 1 + e^{2x}$
$$y = x + \frac{1}{2}e^{2x} + c$$

Separable Equations

Separable Differential Equation

A first-order ODE of the form

$$\frac{dy}{dx} = g(x)\, h(y)$$

is called a **separable equation —** variables can be separated onto opposite sides.

Method of solution:

1. Divide by $h(y)$ to separate: $\dfrac{dy}{h(y)} = g(x)\, dx$
2. Integrate both sides: $\displaystyle\int \frac{dy}{h(y)} = \int g(x)\, dx$
3. Solve (implicitly or explicitly) for y.

Integration Step

Starting from

$$p(y)\,dy = g(x)\,dx, \qquad p(y) = \frac{1}{h(y)}$$

Integrating:

$$\int p(y)\,dy = \int g(x)\,dx \qquad \Longrightarrow \qquad H(y) = G(x) + c$$

- $H(y)$ and $G(x)$ are antiderivatives.
- The result is usually in **implicit form**; solve for y where possible.

Example 1: Separable DE

Solve

$$(1+x)\,dy - y\,dx = 0$$

Divide by $(1+x)y$:

$$\frac{dy}{y} = \frac{dx}{1+x}$$

Integrate both sides:

$$\ln|y| = \ln|1+x| + c$$

$$\boxed{y = c(1+x)}$$

Example 2: IVP

Solve

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad y(4) = -3$$

Separate variables:

$$y\,dy = -x\,dx$$

Integrate:

$$\frac{y^2}{2} = -\frac{x^2}{2} + c \qquad \Rightarrow \qquad x^2 + y^2 = c$$

Apply $y(4) = -3$: $\quad 16 + 9 = 25$, so

$$\boxed{x^2 + y^2 = 25}$$

The solution curve is the lower semicircle $y = -\sqrt{25 - x^2}$.

Example 3: IVP

Solve

$$\frac{dy}{dx} = 2y, \qquad y(0) = 5$$

Separate:

$$\frac{1}{y}\,dy = 2\,dx$$

Integrate:

$$\ln|y| = 2x + C \qquad \Rightarrow \qquad y = Ce^{2x}$$

Apply $y(0) = 5$: $\quad C = 5$

$$\boxed{y = 5e^{2x}}$$

Example 4: Logistic Growth

Solve $\dfrac{dy}{dx} = y(1-y)$.
Use partial fractions to separate:

$$\frac{dy}{y(1-y)} = dx \qquad \Rightarrow \qquad \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = dx$$

Integrate:

$$\ln|y| - \ln|1-y| = x + C \qquad \Rightarrow \qquad \frac{y}{1-y} = Ce^x$$

Solving for y:

$$y = \frac{Ce^x}{1+Ce^x}$$

Example 4: Logistic Growth — IVP

Apply the initial condition $y(0) = \frac{1}{2}$:

$$\frac{1}{2} = \frac{C}{1+C} \quad \Rightarrow \quad 1 + C = 2C \quad \Rightarrow \quad C = 1$$

Final solution:

$$\boxed{y = \frac{e^x}{1+e^x}}$$

- As $x \to \infty$, $y \to 1$ (carrying capacity).
- As $x \to -\infty$, $y \to 0$.
- The population grows from near zero and saturates at 1.

Losing a Solution

Dividing by $h(y)$ during separation may eliminate constant solutions where $h(y) = 0$.

Example: $\dfrac{dy}{dx} = y^2 - 4 = (y-2)(y+2)$

Constant solutions (set $h(y) = 0$): $y = 2$ and $y = -2$.

After separation and integration the general solution is:

$$y = 2\,\frac{1 + ce^{4x}}{1 - ce^{4x}}$$

- $y = 2$ is recovered when $c = 0$.
- $y = -2$ **cannot** be recovered — it is a **singular solution**.

Always Check for Lost Solutions

Rule

Before separating $\dfrac{dy}{dx} = g(x)\,h(y)$, set $h(y) = 0$ and solve.
Each root is a **constant solution** that may be lost after dividing by $h(y)$.

In the example $\dfrac{dy}{dx} = (y-2)(y+2)$:

- $h(y) = (y-2)(y+2) = 0$ gives $y = 2$ and $y = -2$.
- After separation, $y = 2$ can be recovered; $y = -2$ is permanently lost.
- Always verify whether lost solutions satisfy the original equation.

Integral Form of Solutions

For the IVP

$$\frac{dy}{dx} = g(x), \qquad y(x_0) = y_0$$

the solution is

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\, dt$$

Example: Solve $\dfrac{dy}{dx} = e^{-x^2}$, $y(2) = 6$

$$y(x) = 6 + \int_{2}^{x} e^{-t^2}\, dt$$

- The integral cannot be expressed in elementary functions.
- The solution is valid and exact — it just remains in integral form.

Thank you :)