CALCULUS

Lecture 08

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Waves Are Everywhere

Many natural phenomena behave like waves:

- Sound waves in music
- Vibrations of a guitar string
- Heat flow in a metal rod
- Signal transmission in communication systems

In many situations, complicated patterns arise from combining simple waves.

Key Question:
Given a function $f(x)$, can we build it using simple trigonometric waves such as

$$\sin x, \ \sin 2x, \ \sin 3x, \ \cos x, \ \cos 2x, \ldots ?$$

The Idea of Decomposition

The central idea is to express a complicated function as a sum of simple waves.

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \cdots$$

Each term represents a wave with a different frequency.
The coefficients a_n and b_n determine how much of each wave appears in the function.
This representation is called a **Fourier series**.

The Fourier Idea

In the early 1800s, the French mathematician

Joseph Fourier (1768–1830)

proposed that many functions can be expressed as sums of trigonometric waves.

We will study three key ideas behind this theory:

- Inner products of functions
- Orthogonal functions
- Fourier series expansions

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Orthogonal Vectors

Two vectors are called **orthogonal** if they are perpendicular.
Example in $\mathbb{R}^2$:

$$u = (1, 0), \qquad v = (0, 1)$$

These vectors form a right angle.
We can detect this using the **dot product**:

$$u \cdot v = 1 \cdot 0 + 0 \cdot 1 = 0$$

Key idea:

$$u \cdot v = 0 \quad \Longrightarrow \quad u \text{ and } v \text{ are orthogonal}$$

Why Orthogonality Matters

Orthogonal vectors behave like independent directions.
Example:

$$(3,4) = 3(1,0) + 4(0,1)$$

The coefficients are easy to find because the vectors are orthogonal.
Orthogonality allows us to break complicated vectors into simple pieces.
Question:
Can we do something similar for functions?

From Vectors to Functions

Vectors:

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

This is a **sum of products**.

For functions, we replace the sum with an **integral**.

For functions $f(x)$ and $g(x)$ on $[a, b]$:

$$(f, g) = \int_a^b f(x) g(x) \, dx$$

This plays the same role as the dot product.

Orthogonal Functions

Two functions f and g are called **orthogonal** on $[a, b]$ if

$$\int_a^b f(x)g(x)\,dx = 0$$

Example:

$$f(x) = x^2, \qquad g(x) = x^3$$

on the interval $[-1, 1]$.

$$\int_{-1}^1 x^2 x^3\,dx = \int_{-1}^1 x^5\,dx = 0$$

Hence x^2 and x^3 are orthogonal on $[-1, 1]$.

Orthogonal Sets of Functions

A set of functions

$$\{f_0(x), f_1(x), f_2(x), \dots\}$$

is called **orthogonal** if

$$\int_a^b f_m(x) f_n(x)\, dx = 0 \quad \text{whenever } m \neq n$$

Important example:

$$\{1, \cos x, \cos 2x, \cos 3x, \dots\}$$

is orthogonal on $[-\pi, \pi]$.
These functions will form the building blocks of Fourier series.

Breaking Functions into Pieces

Suppose we have an orthogonal set

$$\{f_0(x), f_1(x), f_2(x), \ldots\}$$

We try to represent a function as

$$f(x) = c_0 f_0(x) + c_1 f_1(x) + c_2 f_2(x) + \cdots$$

Orthogonality allows us to compute each coefficient separately.
This idea leads to **Fourier series**.

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Trigonometric Orthogonal Functions

Orthogonal functions satisfy $\int_a^b f_m(x) f_n(x)\, dx = 0 \quad (m \neq n)$

On the interval $[-\pi, \pi]$, the following functions form an orthogonal set:

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

For example when $m \neq n$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = 0 \quad \int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0$$

These trigonometric functions will serve as the building blocks of Fourier series.

Fourier Series

For a function defined on $(-\pi, \pi)$ we write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

This expansion is called the **Fourier series** of $f(x)$.
Interpretation:

- $\sin(nx)$ and $\cos(nx)$ represent different frequencies
- a_n and b_n measure how much of each frequency appears

How the Coefficients Are Found

To determine the coefficients, multiply the series by $\cos(mx)$ and integrate.

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$$

Because of orthogonality,

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0 \quad (m \neq n)$$

All terms vanish except the one containing $\cos(mx)$.
This isolates the coefficient a_m.
A similar idea gives the formula for b_n.

Fourier Coefficients

The coefficients are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Each coefficient measures the contribution of that trigonometric wave.

PARTIAL SUMS

Define the **partial sum**

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

$S_N(x)$ approximates the function $f(x)$.

As N increases:

- more frequencies are included
- the approximation improves

Periodic Extension

Fourier series are naturally periodic.

If $f(x)$ is defined on $(-\pi, \pi)$, the Fourier series represents a function satisfying

$$f(x + 2\pi) = f(x)$$

Thus the series actually represents the **periodic extension** of $f(x)$.

Graphically:

- The function repeats every 2π
- The series matches the function on $(-\pi, \pi)$

Convergence of Fourier Series

If $f(x)$ is

- piecewise continuous
- piecewise smooth

then the Fourier series converges to

$$f(x)$$

at points where f is continuous.

At a jump discontinuity, the series converges to

$$\frac{f(x^+) + f(x^-)}{2}$$

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Symmetry of Functions

Some functions have symmetry.

Even functions: $f(-x) = f(x)$
Examples: $x^2, \quad \cos x$
Graphically:

- Symmetric about the y-axis

Odd functions: $f(-x) = -f(x)$
Examples: $x^3, \quad \sin x$
Graphically:

- Symmetric about the origin

Symmetry and Integrals

On a symmetric interval $[-a, a]$:

If $f(x)$ is **even**

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

If $f(x)$ is **odd**

$$\int_{-a}^{a} f(x)\,dx = 0$$

This symmetry will greatly simplify the computation of Fourier co-efficients.

Symmetry of Sine and Cosine

Trigonometric functions also have symmetry.

$$\cos(-x) = \cos(x)$$

So cosine is an **even function**.

$$\sin(-x) = -\sin(x)$$

So sine is an **odd function**.
This symmetry determines which terms appear in a Fourier series.

Symmetry in Fourier Series

Recall the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

Because

- cosine is even
- sine is odd

The symmetry of $f(x)$ determines which coefficients vanish.

Even Functions and Cosine Series

Suppose $f(x)$ is even.
Then

$$b_n = 0$$

because $f(x)\sin(nx)$ becomes an odd function.
The Fourier series reduces to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

This is called the **Fourier cosine series**.

Cosine Series Coefficients

For an even function

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

Advantages:

- Only cosine terms appear
- Integration only over half the interval

Odd Functions and Sine Series

Suppose $f(x)$ is odd.
Then

$$a_n = 0$$

The Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

This is called the **Fourier sine series**.

Sine Series Coefficients

For an odd function

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

Advantages:

- Only sine terms appear
- Integration only over $[0, \pi]$

Half-Range Expansions

Sometimes a function is defined only on $(0, L)$
To apply Fourier series we extend it to $(-L, L)$.
Two common choices:

- **Even extension**

$$f(-x) = f(x)$$

  Produces a cosine series.

- **Odd extension**

$$f(-x) = -f(x)$$

  Produces a sine series.

Why Half-Range Expansions Matter

Half-range expansions appear naturally in many applications.
Examples:

- Heat equation
- Vibrating strings
- Boundary value problems

Boundary conditions often force solutions to contain only

- sine terms, or
- cosine terms.

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Example 1: Fourier Series of $f(x) = x^2$

Find the Fourier series of

$$f(x) = x^2$$

on $(-\pi, \pi)$.
Observe that

$$f(-x) = (-x)^2 = x^2 = f(x)$$

Thus $f(x)$ is **even**.
Therefore

$$b_n = 0$$

and the Fourier series contains only cosine terms.

Computing the Coefficients

Since $f(x)$ is even,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$$

Hence

$$\frac{a_0}{2} = \frac{\pi^2}{3}$$

Computing a_n

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx)\, dx$$

Using integration by parts,

$$a_n = \frac{4(-1)^n}{n^2}$$

Thus the Fourier series becomes

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

Example 2: Fourier Series of $f(x) = x$

Find the Fourier series of

$$f(x) = x$$

on $(-\pi, \pi)$.
Observe

$$f(-x) = -x = -f(x)$$

Thus $f(x)$ is **odd**.
Therefore

$$a_n = 0$$

and only sine terms appear.

COMPUTING b_n

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx$$

Using symmetry,

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx)\, dx$$

Integrating by parts gives

$$b_n = \frac{2(-1)^{n+1}}{n}$$

Fourier Series of x

The Fourier series is

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

This series represents the periodic extension of x.

Example 3: Half-Range Cosine Series

Find the Fourier cosine series for

$$f(x) = x$$

on $(0, \pi)$.
We extend $f(x)$ to an even function:

$$f(-x) = f(x)$$

Thus the series contains only cosine terms.
Even extension becomes $|x|$

Computing Coefficients

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) \, dx$$

Integration by parts gives

$$a_n = \frac{2(-1)^n - 2}{\pi n^2}$$

Cosine Series

The Fourier cosine series for $f(x) = x$ on $(0, \pi)$ is

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos\big((2k+1)x\big)}{(2k+1)^2}$$

Valid for

$$0 < x < \pi$$

This series represents the **even periodic extension** of $f(x)$.

Thank you :)