CALCULUS

Lecture 07

Anshid Aboobacker

Basic Integration Formulas

1. $\int k\,dx = kx + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$

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Integration by Parts

When to Use It

Used for integrals of the form $\int f(x)g(x)\,dx$ when:

- One function becomes simpler when differentiated
- The other is easy to integrate

Product Rule for derivative

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Rearranging gives the integration formula:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Integration by Parts Formula

Differential Form (Easier to Remember)

$$\int u\,dv = uv - \int v\,du$$

How to Choose *u* and *dv*

- Choose u so that differentiating makes it simpler
- Choose dv so it is easy to integrate
- The new integral $\int v\,du$ must be easier than the original

Common Good Choices for *u*

Inverse trig $\rightarrow$ Logarithmic $\rightarrow$ Algebraic $\rightarrow$ Trigonometric $\rightarrow$ Exponential (ILATE)

Examples

Example 1: $\int x \cos x \, dx$

$$u = x, \quad dv = \cos x \, dx \implies du = dx, \quad v = \sin x$$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$$

Example 2: $\int \ln x \, dx$
Rewrite as: $\int \ln x \cdot 1 \, dx$

$$u = \ln x, \quad dv = dx \implies du = \frac{1}{x} dx, \quad v = x$$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx = x \ln x - x + C$$

Practice Problems

Evaluate the following:

1. $$\int x^2 e^x \, dx$$
2. $$\int x e^{-x} \, dx$$
3. $$\int x^3 \sin x \, dx$$
4. $$\int e^x \cos x \, dx$$
5. $$\int_0^4 x e^{-x} \, dx$$

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Trigonometric Integrals

We evaluate integrals involving powers and products of

$$\sin x, \cos x, \tan x, \sec x$$

Main Idea

Use identities to rewrite the integrand into a simpler form.

Three Major Categories

1. Powers of $\sin x$ and $\cos x$
2. Powers of $\tan x$ and $\sec x$
3. Products like $\sin(mx)\cos(nx)$

Decision Rules

1. Powers of $\sin^m x \cos^n x$

- If m is odd:
  - Save one $\sin x$
  - Convert $\sin^2 x = 1 - \cos^2 x$
  - Substitute $u = \cos x$
- If n is odd:
  - Save one $\cos x$
  - Convert $\cos^2 x = 1 - \sin^2 x$
  - Substitute $u = \sin x$
- If both even:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Decision Rules

2. Powers of $\tan x$ **and** $\sec x$

Use:

$$\tan^2 x = \sec^2 x - 1, \quad \sec^2 x = \tan^2 x + 1$$

3. Products $\sin(mx)\cos(nx)$

Use product-to-sum identities:

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

Examples

$$\int \sin^3 x \cos^2 x \, dx$$

m is odd $\rightarrow$ save one $\sin x$

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

Let $u = \cos x$

$$= \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

Examples

$$\int \tan^4 x \, dx$$

$$= \int \tan^2 x(\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$$

Let $u = \tan x$

$$= \frac{\tan^3 x}{3} - \tan x + x + C$$

Practice Problems

1. $$\int \sin^5 x \cos^2 x \, dx$$
2. $$\int \cos^5 x \, dx$$
3. $$\int \sin^2 x \cos^4 x \, dx$$
4. $$\int \sec^4 x \, dx$$
5. $$\int \sec^3 x \, dx$$
6. $$\int \sin 3x \cos 5x \, dx$$
7. $$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx$$

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Trigonometric Substitution

Trigonometric substitution is used for integrals containing expressions like

$$\sqrt{a^2 + x^2}, \quad \sqrt{a^2 - x^2}, \quad \sqrt{x^2 - a^2}$$

The idea is to replace x by a trigonometric function so that the expression under the square root becomes simpler.

Key Substitutions

$$x = a \tan\theta, \quad x = a \sin\theta, \quad x = a \sec\theta$$

These substitutions come from right–triangle identities.

Choosing the Right Substitution

Expression in the integral	Substitution
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$

General Procedure

1. Substitute x using the appropriate trigonometric function.
2. Compute dx.
3. Simplify the square root using trig identities.
4. Integrate the resulting trigonometric expression.
5. Use a reference triangle to convert back to x.

EXAMPLES

Solve: $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

Use substitution

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta$$

$$\sqrt{4+x^2} = \sqrt{4(1+\tan^2\theta)} = 2\sec\theta$$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

Back-substitute:

$$= \ln\left|x + \sqrt{4+x^2}\right| + C$$

Examples

Solve: $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

Use substitution

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\, d\theta$$

$$\sqrt{9-x^2} = 3\cos\theta$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin\theta)^2 3\cos\theta\, d\theta}{3\cos\theta} = 9\int \sin^2\theta\, d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$$

Back-substitute:

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{9-x^2} + C$$

Practice Problems

Evaluate the following using trigonometric substitution.

1. $$\int \frac{dx}{\sqrt{9+x^2}}$$
2. $$\int \frac{dx}{\sqrt{16-x^2}}$$
3. $$\int \frac{dx}{\sqrt{x^2-25}}$$
4. $$\int \frac{x^2}{\sqrt{9-x^2}}dx$$
5. $$\int \frac{dx}{\sqrt{4x^2-9}}, \quad x > \frac{3}{2}$$
6. $$\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$$

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Integration of Rational Functions

A rational function has the form $\dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials.

Idea of Partial Fractions

Rewrite the rational function as a sum of simpler fractions that are easy to integrate.

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

Then integrate term by term.

$$\int \frac{5x-3}{x^2-2x-3}dx = 2\ln|x+1| + 3\ln|x-3| + C$$

Case 1: Distinct Linear Factors

If $g(x) = (x - r_1)(x - r_2)\ldots(x - r_n)$ then,

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \ldots + \frac{A_n}{x - r_n}$$

Example 1

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} dx = \int \left(\frac{3/4}{x-1} + \frac{1/2}{x+1} - \frac{1/4}{x+3} \right) dx$$

$$= \frac{3}{4} \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln|x+3| + C$$

Example 2

$$\int \frac{5x-3}{(x+1)(x-3)} dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3} \right) dx$$

$$= 2 \ln|x+1| + 3 \ln|x-3| + C$$

CASE 2: REPEATED LINEAR FACTORS

If $g(x) = (x - r)^m$ then the partial fraction form is

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

Example 1

$$\int \frac{6x+7}{(x+2)^2} dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2} \right) dx$$

$$= 6 \ln |x+2| + \frac{5}{x+2} + C$$

Example 2

$$\int \frac{x-1}{(x+1)^3} dx = \int \left(\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3} \right) dx$$

$$= -\frac{1}{x+1} + \frac{1}{(x+1)^2} + C$$

Case 3: Irreducible Quadratic Factors

If the denominator contains $x^2 + px + q$ that has no real roots, we use $\dfrac{Ax+B}{x^2+px+q}$; If the quadratic repeats, $\dfrac{Ax+B}{x^2+px+q} + \dfrac{Cx+D}{(x^2+px+q)^2}$

Example 1: $\int \frac{3x+5}{x^2+1}\,dx$

$\frac{3x+5}{x^2+1} = \frac{Ax+B}{x^2+1} \implies A = 3,\ B = 5$

$\int \frac{3x+5}{x^2+1}dx = \int \frac{3x}{x^2+1}dx + \int \frac{5}{x^2+1}dx = \frac{3}{2}\ln(x^2+1) + 5\tan^{-1} x + C$

Example 2: $\int \frac{2x+1}{(x^2+1)^2}dx$

$\frac{2x+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \implies 2x+1 = (Ax+B)(x^2+1) + (Cx+D)$

$2x+1 = Ax^3 + Bx^2 + (A+C)x + (B+D) \implies A = B = 0,\ C = 2,\ D = 1$

$\int \frac{2x+1}{(x^2+1)^2}dx = \int \frac{2x}{(x^2+1)^2}dx + \int \frac{1}{(x^2+1)^2}dx = -\frac{1}{x^2+1} + \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x + C$

Practice Problems

Evaluate:

1. $$\int \frac{5x-3}{(x+1)(x-3)}dx$$
2. $$\int \frac{6x+7}{(x+2)^2}dx$$
3. $$\int \frac{x}{(x^2+1)^2}dx$$
4. $$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx$$
5. $$\int \frac{x+4}{x(x-2)(x+5)}dx$$
6. $$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx$$

Thank you :)