CALCULUS

Lecture 06

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Why Do We Need Integration?

Classical geometry gives formulas for:

- Areas of triangles
- Volumes of spheres and cones

But what about:

- Regions with curved boundaries?
- Quantities that change continuously?

How do we compute area under a curve?

To approximate the area under $f(x)$ on $[a, b]$:

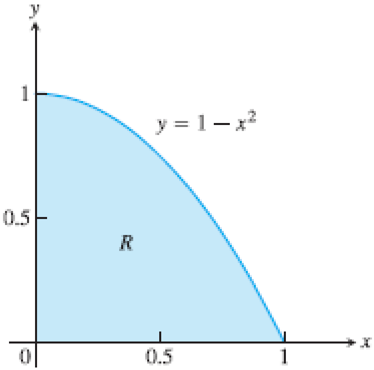
1. Divide $[a, b]$ into small subintervals
2. Build rectangles over each piece
3. Add their areas

Curved region area = Sum of areas of simple rectangles

Example: Area Under $y = 1 - x^2$ on $[0, 1]$

Region:

- Above x-axis
- Below $y = 1 - x^2$
- Between $x = 0$ and $x = 1$



Lower and Upper Approximations

For each subinterval:

- Use the **minimum value** of $f \Rightarrow$ Lower sum
- Use the **maximum value** of $f \Rightarrow$ Upper sum

$$\text{Lower Sum} \leq \text{True Area} \leq \text{Upper Sum}$$

Refining the Approximation

If we:

- Increase the number of rectangles
- Make each rectangle thinner

Then:

- Lower sums increase
- Upper sums decrease

When the upper and lower sums approach the same value:

That common value = Area under the curve

From Numbers to Continuous Change

Suppose you record temperatures: $70, \ 72, \ 75, \ 71$
Then, average temperature $= \frac{70+72+75+71}{4} = 72$

Average = total quantity divided by number of measurements.

Now suppose temperature varies continuously on $[a, b]$.

- There are infinitely many values.
- We cannot directly "add them up".
- Instead, we approximate the total using rectangles:

$$\text{Total} \approx f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$$

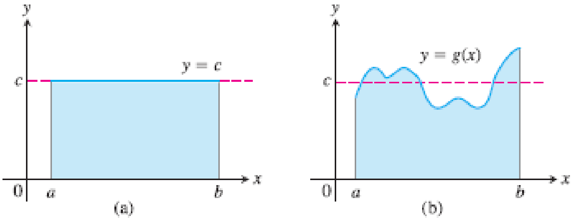
Each term represents a small rectangular area. The rectangular sum approximates the **area under the curve**.
So the continuous average should be:

$$\text{Average value} \approx \frac{\text{Area under } f}{b - a}$$

GEOMETRIC MEANING

If $f(x) = c$ is constant:

$$\text{Area} = c(b-a) \quad \Rightarrow \quad \text{Average} = c$$



The average value is the height of a rectangle with base $b - a$ having the same area as the region under f.

Why Sigma Notation?

Finite sums often have many terms: $a_1 + a_2 + \cdots + a_n$
Writing them explicitly is inefficient.

Sigma Notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

k = index; Lower number = start; Upper number = end

Examples:

$$\sum_{k=1}^{11} k^2 = 1^2 + 2^2 + \cdots + 11^2 \qquad \sum_{i=1}^{100} f(i) = f(1) + \cdots + f(100)$$

The lower limit need not be 1: $\sum_{k=4}^{5} \frac{k^2}{k-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1} = \frac{16}{3} + \frac{25}{4}$

Algebra Rules for Finite Sums

Linearity Rules

$$\sum_{k=n}^{m}(a_k \pm b_k) = \sum_{k=n}^{m} a_k \pm \sum_{k=n}^{m} b_k$$

$$\sum_{k=n}^{m} ca_k = c\sum_{k=n}^{m} a_k \qquad \sum_{k=1}^{n} c = nc$$

Formulas for Special Sums

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Riemann Sum

Let f be defined on $[a, b]$.
Divide $[a, b]$ into n equal parts:

$$\Delta x = \frac{b - a}{n}$$

Choose one point c_k in each subinterval.

The corresponding rectangular area sum is:

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x$$

This is called a **Riemann sum**.

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Definite Integral

If the limit exists as $n \to \infty$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

then this limit is called the

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x}$$

Components:

- a, b — limits of integration
- $f(x)$ — integrand
- dx — variable of integration

Interpretation: Integral = limit of Riemann sums.

When is a Function Integrable?

- If f is continuous on $[a, b]$, then f is integrable.
- If f has finitely many jump discontinuities on $[a, b]$, it is integrable.
- Most elementary functions are integrable.

Example of a Nonintegrable Function

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases} \quad \text{on } [0, 1]$$

Every subinterval contains rational and irrational numbers
Upper sum = 1, Lower sum = 0
No unique limit $\Rightarrow$ Not integrable.

PROPERTIES OF THE DEFINITE INTEGRAL

Assume f and g are integrable on $[a, b]$ and let $a < c < b$

1. Orientation $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. Zero Width $\int_a^a f(x)\,dx = 0$

3. Constant Multiple $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4. Sum Rule $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

5. Additivity Over Intervals $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

6. Min–Max Inequality
If $m \leq f(x) \leq M$ on $[a, b]$, $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

7. Domination
If $f(x) \leq g(x)$ on $[a, b]$, $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

Area Under the Curve

If $f(x) \geq 0$ and is integrable on $[a, b]$, then the area under the curve $y = f(x)$ from a to b is

$$A = \int_a^b f(x)\, dx.$$

Example: Compute $\int_0^b x^2\, dx$ Using Riemann Sums

- Partition $[0, b]$ into n equal parts: $\Delta x = \frac{b}{n}$
- Choose right endpoints: $x_i = \frac{ib}{n}$

Riemann sum: $\sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \frac{b}{n} = \frac{b^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{b^3}{6} \frac{n(n+1)(2n+1)}{n^3}$

Taking limit as $n \to \infty$: $\int_0^b x^2 dx = \frac{b^3}{3}$

Special Formulas and Signed Area

From similar computations:

$$\int_a^b c\,dx = c(b-a)$$

$$\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$$

Signed Area

- If $f(x) > 0$: area above x-axis
- If $f(x) < 0$: area below x-axis
- Integral gives net signed area

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Mean Value Theorem for Definite Integrals

Theorem

If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

- The average value is actually attained.
- From Max–Min inequality: $\min f \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \max f$
- By Intermediate Value Theorem: Every value between $\min f$ and $\max f$ is assumed.
- Therefore: $f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$ for some c.

Example

If f is continuous on $[a, b]$, $a \neq b$, and

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution Idea:

- Average value:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = 0$$

- By Mean Value Theorem:

$$f(c) = 0 \quad \text{for some } c \in [a, b].$$

DEFINING AN INTEGRAL FUNCTION

Definition

Let f be integrable on $[a, b]$. Define

$$F(x) = \int_a^x f(t)\,dt.$$

- x is the upper limit.
- $F(x)$ represents area from a to x.
- F is a new function defined by an integral.

Geometric Insight

- Consider difference quotient:

$$\frac{F(x+h)-F(x)}{h}$$

- Numerator:

$$F(x+h)-F(x)=\int_x^{x+h} f(t)\,dt$$

- For small h:
    - Area $\approx f(x)\cdot h$
- Therefore:

$$\frac{F(x+h)-F(x)}{h}\approx f(x)$$

Fundamental Theorem of Calculus 1

Theorem

If f is continuous on $[a, b]$ and

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$F'(x) = f(x).$$

- Integration followed by differentiation returns the original function.

Fundamental Theorem of Calculus 2

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx == F(x)\Big|_a^b = F(b) - F(a).$$

- Definite integrals can be computed using antiderivatives.
- No need for Riemann sums.

To evaluate $\int_a^b f(x)\,dx$:

1. Find an antiderivative F of f.
2. Compute $F(b) - F(a)$.

Definite integral reduces to subtraction.

EXAMPLES

1) $\int_0^{\pi} \cos x \, dx$
$\int \cos x \, dx = \sin x$
$\implies \int_0^{\pi} \cos x \, dx = \sin \pi - \sin 0 = 0 - 0 = 0$

2) $\int_{-\pi/4}^{0} \sec x \tan x \, dx$
$\int \sec x \tan x \, dx = \sec x$
$\implies \int_{-\pi/4}^{0} \sec x \tan x \, dx = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$

3) $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$
Antiderivative: $x^{3/2} + \frac{4}{x}$
$\implies \left[x^{3/2} + \frac{4}{x}\right]_1^4 = (8 + 1) - (1 + 4) = 4$

Net Change Theorem

If $F'(x) = f(x)$, then $\int_a^b F'(x)\,dx = F(b) - F(a)$.

- Integral of a rate of change
- Equals net change in the quantity

Theorem

The net change in $F(x)$ over $[a, b]$ is

$$F(b) - F(a) = \int_a^b F'(x)\,dx.$$

- Integrating velocity gives displacement.
- Integrating marginal cost gives total cost change.

Integration and Differentiation

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

- Differentiate after integrating $\rightarrow$ original function.
- Integrate after differentiating $\rightarrow$ net change.

Example

For $f(x) = x^2 - 4$ on $[-2, 2]$:

$$\int_{-2}^{2} f(x)\, dx = -\frac{32}{3}$$

Mirror function:

$$g(x) = 4 - x^2$$

$$\int_{-2}^{2} g(x)\, dx = \frac{32}{3}$$

- Same geometric area
- Different signs

Example

$$f(x) = \sin x, \quad [0, 2\pi]$$

$$\int_0^{2\pi} \sin x \, dx = 0$$

Total area:

$$\int_0^{\pi} \sin x \, dx = 2$$

$$\int_{\pi}^{2\pi} \sin x \, dx = -2$$

$$\text{Total Area} = 4$$

Practice

1) Find $\frac{d}{dx}\left(\int_2^x (3t^2 - 4t + 1)\,dt\right)$.
Solution: By FTC Part 1:

$$\frac{d}{dx}\int_2^x f(t)\,dt = f(x) \implies \text{Ans} = 3x^2 - 4x + 1.$$

2) Find $\frac{d}{dx}\left(\int_0^{x^3} \cos t\,dt\right)$.
Solution:
Let $F(u) = \int_0^u \cos t\,dt.$ $\implies$ $\frac{d}{dx}F(x^3) = F'(x^3) \cdot 3x^2$
$F'(u) = \cos u$
$\implies$ Ans $= \cos(x^3) \cdot 3x^2$.

Practice

3) Evaluate $\int_1^3 (2x^3 - 5x)\,dx$.
Solution:
Antiderivative: $\frac{1}{2}x^4 - \frac{5}{2}x^2$
$\implies \left[\frac{1}{2}x^4 - \frac{5}{2}x^2\right]_1^3 = 18 - (-2) = 20.$

4) Evaluate $\int_0^{\pi/2} \sin x\,dx$.
Solution: $\int \sin x\,dx = -\cos x$
$[-\cos x]_0^{\pi/2} = 0 - (-1) = 1.$

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Indefinite Integral: Definition

The **indefinite integral** of a function f with respect to x is the set of all antiderivatives of f, denoted by $\int f(x)\,dx$.

Meaning: If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant.
Any two antiderivatives of f differ by a constant.

- A **definite integral** $\int_a^b f(x)\,dx$ is a **number**.
- An **indefinite integral** $\int f(x)\,dx$ is a **function** plus a constant C.

Example

Problem

Find

$$\int (x^3 + x)^5 (3x^2 + 1)\, dx.$$

Solution

Let $u = x^3 + x$, then $du = (3x^2 + 1)\, dx$.

$$\int u^5\, du = \frac{u^6}{6} + C$$

$$= \frac{(x^3 + x)^6}{6} + C.$$

Example

Problem

$$\int \sqrt{2x+1}\,dx.$$

Solution

Let $u = 2x + 1$, so $du = 2\,dx$.

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C.$$

The Substitution Rule

Theorem (Substitution Rule)

If $u = g(x)$ is differentiable and f is continuous, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Substitution Method (Steps)

1. Let $u = g(x)$.
2. Compute $du = g'(x)\,dx$.
3. Rewrite integral in terms of u.
4. Integrate.
5. Substitute back.

Example

Problem

$$\int \sec^2(5t+1)\cdot 5\,dt.$$

Solution

Let $u = 5t+1$, $du = 5\,dt$.

$$\int \sec^2 u\,du = \tan u + C$$

$$= \tan(5t+1) + C.$$

Example

Problem

$$\int \cos(7\theta + 3)\, d\theta.$$

Solution

Let $u = 7\theta + 3$, $du = 7\, d\theta$.

$$= \frac{1}{7} \int \cos u\, du = \frac{1}{7} \sin u + C$$

$$= \frac{1}{7} \sin(7\theta + 3) + C.$$

Example

Problem

$$\int x^2 \sin(x^3)\, dx.$$

Solution

Let $u = x^3$, $du = 3x^2 dx$.

$$= \frac{1}{3}\int \sin u\, du = -\frac{1}{3}\cos u + C$$

$$= -\frac{1}{3}\cos(x^3) + C.$$

Example

Problem

$$\int x\sqrt{2x+1}\,dx$$

Solution

Let $u = 2x + 1$, so $du = 2dx$ and $x = \frac{u-1}{2}$.
Then

$$\int x\sqrt{2x+1}\,dx = \frac{1}{4}\int (u-1)u^{1/2}\,du.$$

$$= \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du. = \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C.$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C.$$

Example

Problem: $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

Solution 1 Let $u = z^2 + 1$, $du = 2z\,dz$.

$$= \int u^{-1/3}\,du = \frac{u^{2/3}}{2/3} + C = \frac{3}{2}(z^2+1)^{2/3} + C.$$

Solution 2 Let $u = \sqrt[3]{z^2+1}$. Then $u^3 = z^2 + 1$ and $3u^2 du = 2zdz$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2}{u} du = 3\int u\,du = \frac{3}{2}u^2 + C.$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C.$$

Example

Use $\sin^2 x = \frac{1-\cos 2x}{2}$
(a) $\int \sin^2 x \, dx$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

Use $\cos^2 x = \frac{1+\cos 2x}{2}$
(b) $\int \cos^2 x \, dx$

$$= \frac{1}{2} \int (1 + \cos 2x) \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

Thank you :)