CALCULUS

Lecture 05

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Applications of Derivatives

Applications of Derivatives

- Find maximum and minimum values
- Analyze shapes of graphs (increasing/decreasing, concavity)
- Recover a function from its derivative
- Key theoretical tool: Mean Value Theorem

Absolute (Global) Extrema

Let f have domain D.

- **Absolute Maximum at** c: $f(x) \leq f(c) \quad \forall x \in D$
- **Absolute Minimum at** c: $f(x) \geq f(c) \quad \forall x \in D$

Absolute maxima/minima are called **global extrema**.

Extreme Values Depend on the Domain

Example 1: Trigonometric Functions on $[-\pi/2, \pi/2]$

- $f(x) = \cos x$
  - Max = 1 at $x = 0$ Min = 0 at $x = \pm\pi/2$
- $f(x) = \sin x$
  - Max = 1 Min = -1

Example 2: $f(x) = x^2$

- $(-\infty, \infty)$: Min = 0, no max
- $[0, 2]$: Min = 0, Max = 4
- $(0, 2]$: Max = 4, no min
- $(0, 2)$: No absolute extrema

Conclusion: Closed vs open interval matters.

Extreme Value Theorem

Extreme Value Theorem

If f is continuous on a **closed interval** $[a, b]$, then f attains:

- an absolute minimum value m
- an absolute maximum value M

That is, $m \leq f(x) \leq M \quad$ for all $x \in [a, b]$.

Why Both Conditions Are Essential

- If the interval is not closed $\rightarrow$ extreme value may fail to exist.
- If the function is not continuous $\rightarrow$ extreme value may fail even on $[a, b]$.

Local and Absolute Extrema

Local Maximum

f has a local maximum at c if $f(x) \leq f(c)$ for all x near c.

Local Minimum

f has a local minimum at c if $f(x) \geq f(c)$ for all x near c.

Comparison

- Absolute maximum:
  - Largest value on entire domain
  - Automatically a local maximum
- Local maximum:
  - Largest only in a neighborhood
  - May not be global

First Derivative and Critical Points

First Derivative Theorem

If f has a local maximum or minimum at an interior point c and f' exists at c, then $f'(c) = 0$.

Interpretation At a smooth peak/valley:

- Slope changes sign
- Tangent line is horizontal

Where Can Extreme Values Occur?

1. Interior points where $f'(x) = 0$
2. Interior points where $f'(x)$ does not exist
3. Endpoints

Critical Point

An interior point where $f'(x) = 0 \quad \text{or} \quad f'(x)$ does not exist

Important: A critical point need not be an extremum.
Example: $f(x) = x^3$, $f'(0) = 0$ but no local extremum (point of inflection).

Absolute Extrema on a Closed Interval

Procedure (Continuous f on $[a, b]$)

1. Find all critical points in (a, b).
2. Evaluate f at:
   - Critical points
   - Endpoints a, b
3. Largest value $\rightarrow$ Absolute maximum
4. Smallest value $\rightarrow$ Absolute minimum

Reason: Extreme Value Theorem guarantees existence.

Examples of Absolute Extrema

1. $f(x) = x^2$ **on** $[-2, 1]$

- $f'(x) = 2x \Rightarrow x = 0$
- Evaluate: $f(0) = 0$, $f(-2) = 4$, $f(1) = 1$
- Max = 4, Min = 0

2. $g(t) = 8t - t^4$ **on** $[-2, 1]$

- $g'(t) = 8 - 4t^3 \Rightarrow t = \sqrt[3]{2} > 1$
- No interior critical point
- Evaluate endpoints $\rightarrow$ Max = 7, Min = -32

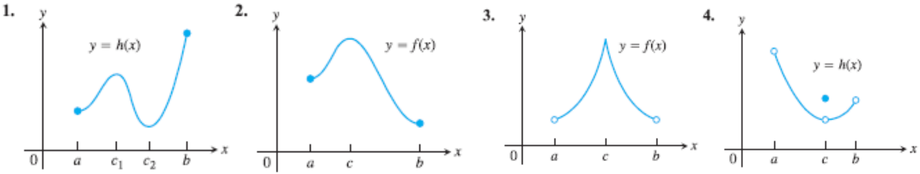
3. $f(x) = x^{2/3}$ **on** $[-2, 3]$

- $f'(x) = \frac{2}{3}x^{-1/3}$ (undefined at 0)
- Check 0 and endpoints
- Min = 0, Max = $3^{2/3}$

Practice: Extreme Values from Graphs

For each graph on $[a, b]$:

- Does an absolute maximum exist?
- Does an absolute minimum exist?
- Where do they occur?
- Does extreme value theorem apply?



Check: Interior peaks/valleys, Endpoints, Closed vs open circles, Continuity.

Practice: Absolute Extrema on Closed Intervals

1. Find critical points in (a, b).
2. Evaluate f at critical points and endpoints.
3. Compare values.

1. $f(x) = \frac{2}{3}x - 5$, $-2 \leq x \leq 3$
2. $f(x) = x^2 - 1$, $-1 \leq x \leq 2$
3. $f(x) = 4 - x^2$, $-3 \leq x \leq 1$
4. $f(x) = -\frac{1}{x^2}$, $0.5 \leq x \leq 2$

Practice: Critical Points

1. $f(x) = x^3 - 8x + 9$
   - Compute $f'(x)$
   - Solve $f'(x) = 0$
   - Any undefined points?
2. $f(x) = \sqrt{x^2 - 1}$
   - Determine domain
   - Compute derivative
   - Where is derivative undefined?

Practice: Concept Check

True or False (justify):

- If $f'(c) = 0$, then f has a local extremum.
- Continuous on $(a, b) \implies$ absolute maximum exists.
- Every absolute maximum is local.

Short Answer:

- Why can a critical point fail to be an extremum?
- Why must we check endpoints?
- Can infinitely many local extrema exist?

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Rolle's Theorem: Intuition and Statement

Motivation

- If a curve starts and ends at the same height,
- It must flatten somewhere in between.

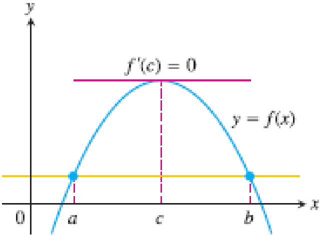
Question: Can a non-constant function have $f'(c) = 0$ somewhere?

Rolle's Theorem

If f is:

- Continuous on $[a, b]$,
- Differentiable on (a, b),
- $f(a) = f(b)$,

then there exists $c \in (a, b)$ such that $f'(c) = 0$.

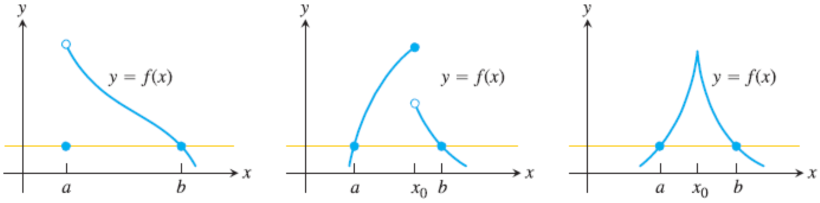


Why Each Hypothesis Is Essential

If any condition fails, the conclusion may fail.

- Not continuous on $[a, b]$ $\rightarrow$ break in graph.
- Not differentiable on (a, b) $\rightarrow$ sharp corner.
- $f(a) \neq f(b)$ $\rightarrow$ no guarantee of flattening.

Conclusion: No horizontal tangent may exist.



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

Application: Uniqueness of a Root

Q: Show that $x^3 + 3x + 1 = 0$ has exactly one real solution.

Define $f(x) = x^3 + 3x + 1$.
Existence: $f(-1) = -3$, $f(0) = 1$
By intermediate value theorem, at least one root in $(-1, 0)$.
Uniqueness: $f'(x) = 3x^2 + 3 > 0$

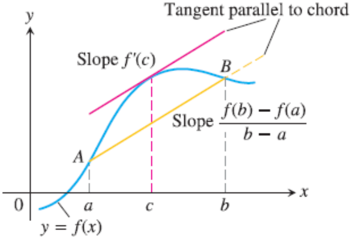
- f strictly increasing.
- Cannot have two roots (Rolle's Theorem).
  If it had two roots, say a and b, then apply Rolle's theorem on intervel $[a, b]$ to get a $c \in [a, b]$ such that $f'(c) = 0$, which is a contradiction.

$\Rightarrow$ Exactly one real solution.

Mean Value Theorem (MVT)

From Rolle to MVT

- Rolle: If $f(a) = f(b) \to$ horizontal tangent exists.
- Question: What if $f(a) \neq f(b)$?
- Intuition:
  - Draw chord joining $A(a, f(a))$ and $B(b, f(b))$.
  - Somewhere, a tangent is parallel to this chord.



Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

RHS = slope of chord, LHS = slope of tangent.

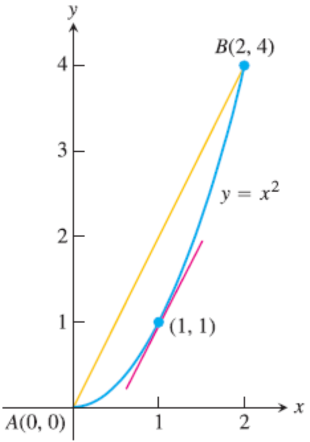
EXAMPLE

$f(x) = x^2$ on $[0, 2]$
Find c satisfying the Mean Value Theorem.

$$\frac{f(2) - f(0)}{2 - 0} = 2.$$

$$f'(x) = 2x.$$

$$2x = 2 \quad \Rightarrow c = 1.$$



Physical Interpretation of MVT

$$\frac{f(b) - f(a)}{b - a} = \text{Average rate of change}$$

$$f'(c) = \text{Instantaneous rate}$$

MVT says: At some time, instantaneous rate equals average rate.

Example (Motion): A car travels 352 ft in 8 sec.
Average velocity:

$$44 \text{ ft/sec.}$$

$$\Rightarrow \exists t \text{ such that } v(t) = 44.$$

Consequences of the Mean Value Theorem

Corollary 1

If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x) = C$.

Corollary 2

If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$.

Interpretation:

- Zero derivative $\implies$ constant function.
- Same derivative $\implies$ functions differ by constant.
- Foundation of indefinite integration.

Example: Recovering a Function from Its Derivative

Q: Find $f(x)$ if $f'(x) = \sin x, \quad f(0) = 2.$

We know:

$$\frac{d}{dx}(-\cos x) = \sin x.$$

So,

$$f(x) = -\cos x + C.$$

Use $f(0) = 2$:

$$-1 + C = 2 \Rightarrow C = 3.$$

$$\Rightarrow f(x) = -\cos x + 3.$$

Practice: Rolle's Theorem

Verify that Rolle's Theorem applies to $f(x) = x^2 - 4x + 3$ on $[1, 3]$. Find all numbers c satisfying the conclusion.

Check Hypotheses

- Polynomial $\Rightarrow$ continuous on $[1, 3]$
- Polynomial $\Rightarrow$ differentiable on $(1, 3)$
- $f(1) = 0$, $f(3) = 0$

All hypotheses satisfied.

Apply Rolle's Theorem

$$f'(x) = 2x - 4$$

Set $f'(c) = 0$: $\quad 2c - 4 = 0 \Rightarrow c = 2$

Practice: Mean Value Theorem

Find all numbers c satisfying the Mean Value Theorem for $f(x) = \sqrt{x}$ on $[1, 4]$.

Hypotheses

- Continuous on $[1, 4]$
- Differentiable on $(1, 4)$

Compute Average Rate: $\frac{f(4)-f(1)}{4-1} = \frac{2-1}{3} = \frac{1}{3}$

Derivative: $f'(x) = \frac{1}{2\sqrt{x}}$

$$\implies \frac{1}{2\sqrt{c}} = \frac{1}{3} \implies c = \frac{9}{4}$$

Practice: Average Speed

A car travels 120 km in 2 hours. Show that at some time during the trip, its speed was exactly 60 km/h.

Let $s(t)$ be distance traveled.
Average speed $= \frac{120}{2} = 60$ km/h.
Since:

- $s(t)$ is continuous,
- $s(t)$ is differentiable,

By Mean Value Theorem:
There exists $c \in (0, 2)$ such that $s'(c) = 60$.

$\boxed{\text{Speed equals average speed at some time}}$

Practice: Inequality via MVT

Use the Mean Value Theorem to show that for $x > 0$, $\ln(1+x) < x$.

Apply MVT to $f(t) = \ln(1 + t)$ on $[0, x]$. There exists $c \in (0, x)$ such that

$$\frac{\ln(1 + x) - \ln(1)}{x} = f'(c) \implies \frac{\ln(1 + x)}{x} = \frac{1}{1 + c}$$

Since $c > 0$: $\frac{1}{1+c} < 1$

Thus:

$$\frac{\ln(1 + x)}{x} < 1 \implies \ln(1 + x) < x$$

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MONOTONICITY

Goal:

- Determine where a function increases/decreases
- Identify local extrema

Monotonicity from Derivative

If f is continuous on $[a, b]$ and differentiable on (a, b):

- If $f'(x) > 0$ for all $x \in (a, b)$, then f is **increasing** on $[a, b]$.
- If $f'(x) < 0$ for all $x \in (a, b)$, then f is **decreasing** on $[a, b]$.

$$\text{Sign of } f' \Longrightarrow \text{Behavior of } f$$

Example: Sign Analysis

Determine the intervals on which the function $f(x) = x^3 - 12x - 5$ is increasing and decreasing.

$f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$
Critical points: $x = -2,\ 2$

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	Increasing	Decreasing	Increasing

Conclusion:

- $f' > 0 \Rightarrow$ slope positive $\Rightarrow$ graph rising
- $f' < 0 \Rightarrow$ slope negative $\Rightarrow$ graph falling
- Increasing on $(-\infty, -2)$ and $(2, \infty)$
- Decreasing on $(-2, 2)$

FIRST DERIVATIVE TEST

First Derivative Test

Let c be a critical point of a continuous function f.

1. If f' changes from positive to negative at c,
   $\Rightarrow$ f has a **local maximum** at c.
2. If f' changes from negative to positive at c,
   $\Rightarrow$ f has a **local minimum** at c.
3. If f' does not change sign,
   $\Rightarrow$ No local extremum at c.

Example: First Derivative Test

Problem. Find the critical points of $f(x) = x^{1/3}(x-4)$. Identify the intervals where f is increasing and decreasing. Find all local and absolute extrema.

Rewrite and Differentiate

$$f(x) = x^{4/3} - 4x^{1/3} \implies f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4(x-1)}{3x^{2/3}}$$

Critical Points: $f'(x) = 0 \Rightarrow x = 1; \quad f'(x)$ undefined at $x = 0$

Sign of f'

- $(-\infty, 0)$: $f' < 0 \Rightarrow$ decreasing
- $(0, 1)$: $f' < 0 \Rightarrow$ decreasing
- $(1, \infty)$: $f' > 0 \Rightarrow$ increasing

Conclusion (First Derivative Test)

- No extremum at $x = 0$ (no sign change)
- Local and absolute minimum at $x = 1$ with $f(1) = -3$

Concavity and the Second Derivative

Direction vs Bending

- Increasing/decreasing $\rightarrow$ direction
- Concavity $\rightarrow$ how the graph bends

Slopes:

- Slopes increasing $\rightarrow$ graph bends upward(concave up)
- Slopes decreasing $\rightarrow$ graph bends downward(concave down)

The graph of f is:

- **Concave up** on an interval I if f' is increasing on I.
- **Concave down** on an interval I if f' is decreasing on I.

Second Derivative Test (Concavity)

Let f be twice differentiable on an interval I.

- If $f''(x) > 0$ on I, graph is concave up.
- If $f''(x) < 0$ on I, graph is concave down.

$f'' > 0$ upward acceleration, $f'' < 0$ downward acceleration.

Points of Inflection

A **point of inflection** is a point where:

- The concavity changes
- Graph changes from concave up to concave down, or vice versa

Necessary condition: $f''(c) = 0 \quad \text{or} \quad f''(c)$ does not exist
But concavity(sign of f'') must change.

Example: $f(x) = x^3$

$$f'(x) = 3x^2, \quad f''(x) = 6x \implies f''(0) = 0$$

Since f'' changes sign at 0: $\quad (0, 0)$ is an inflection point

Second Derivative Test

Suppose $f'(c) = 0$ and $f''(c)$ exists.

1. If $f''(c) > 0$, f has a local minimum at c.
2. If $f''(c) < 0$, f has a local maximum at c.
3. If $f''(c) = 0$, test fails.

Why?

- $f''(c) > 0 \Rightarrow$ concave up
  - Graph shaped like a cup
  - Critical point is minimum
- $f''(c) < 0 \Rightarrow$ concave down
  - Graph shaped like a cap
  - Critical point is maximum

Note:

- Determine sign of f': Increasing / Decreasing intervals
- Determine sign of f'': Concave up / Concave down

Practice: Local Extrema

Find the intervals where $f(x) = x^3 - 12x - 5$ is increasing/decreasing and its local extrema.

$$f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$$

Critical points: $x = -2, \quad x = 2$

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$f'(x)$	$+$	$-$	$+$
Behavior	Inc	Dec	Inc

Local max at $x = -2$, local min at $x = 2$

$$f(-2) = 11, \quad f(2) = -21$$

Practice: Concavity and Inflection

Find intervals of concavity and points of inflection of $f(x) = x^3$

$$f'(x) = 3x^2; \quad f''(x) = 6x$$

- $f''(x) < 0$ for $x < 0 \rightarrow$ concave down
- $f''(x) > 0$ for $x > 0 \rightarrow$ concave up

$$f''(0) = 0$$

Since concavity changes at 0:

$$(0, 0) \text{ is an inflection point}$$

Practice: Second Derivative Test

Classify critical points of: $f(x) = x^4 - 4x^3 + 10$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

Critical points: $x = 0, \quad x = 3$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

$$f''(3) = 36 > 0 \Rightarrow \text{Local Minimum}$$

$$f''(0) = 0 \Rightarrow \text{Test fails}$$

Conclusion:

- Local minimum at $x = 3$
- $x = 0$ requires First Derivative Test

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Antiderivatives: Motivation and Definition

Why Antiderivatives?

- We know how to find the derivative of a function.
- In many applications, we need to reverse this process.
- Example:
  - If velocity is known, how do we find position?
  - If acceleration is known, how do we find velocity?
- This motivates the concept of an **antiderivative**.

Definition

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x) \quad \text{for all } x \in I$.

- The process of finding F from f is called **antidifferentiation**.
- We use capital letters to denote antiderivatives.

Example: Finding Antiderivatives

Find an antiderivative for:

(a) $f(x) = 2x$

(b) $g(x) = \cos x$

(c) $h(x) = 2x + \cos x$

Thinking backward: Which function has this derivative?

(a) $F(x) = x^2$

(b) $G(x) = \sin x$

(c) $H(x) = x^2 + \sin x$

Check: Differentiate each answer.

Antiderivatives Are Not Unique

- If $F(x) = x^2$, then $F'(x) = 2x$.
- But $(x^2 + 1)' = 2x$ also.
- More generally: $\frac{d}{dx}(x^2 + C) = 2x$ for any constant C.

Any two antiderivatives of a function differ by a constant.

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

- This represents a **family of curves**.
- All graphs differ by vertical shifts.

Example: Initial Condition

Find an antiderivative of $f(x) = 3x^2$ such that $F(1) = -1$.

- General antiderivative: $F(x) = x^3 + C$
- $F(1) = 1 + C = -1 \implies C = -2$
- $\implies F(x) = x^3 - 2$

Geometric Insights:

- $F(x) = x^3 + C$ represents infinitely many curves.
- The condition $F(1) = -1$ selects one specific curve.

Engineering Insights:

- The constant C depends on initial conditions.
- Physically, it represents starting position, offset, or baseline value.

Example: General Antiderivatives

Find the general antiderivative of:

(a) $f(x) = x^5$

(b) $g(x) = \dfrac{1}{\sqrt{x}}$

(c) $h(x) = \sin 2x$

(d) $i(x) = \cos \dfrac{x}{2}$

Power Rule for Antiderivatives

For $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

- (a) $\displaystyle\int x^5 dx = \frac{x^6}{6} + C$
- (b) $\displaystyle\int x^{-1/2} dx = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$

Trigonometric Antiderivatives

for $k \neq 0$

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$$

- (c) $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$
- (d) $\int \cos\frac{x}{2}\,dx = 2\sin\frac{x}{2} + C$

LINEARITY RULES

1. **Constant Multiple Rule:**

$$\int kf(x)\,dx = k\int f(x)\,dx$$

2. **Negative Rule:**

$$\int -f(x)\,dx = -\int f(x)\,dx$$

3. **Sum/Difference Rule:**

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

EXAMPLE: ANTIDERIVATIVES

Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x$$

- $\displaystyle\int \frac{3}{\sqrt{x}} dx = 3 \cdot 2\sqrt{x} = 6\sqrt{x}$
- $\displaystyle\int \sin 2x dx = -\frac{1}{2}\cos 2x$

$$\boxed{F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C}$$

PRACTICE

Find the most general antiderivative of:

$$f(x) = 5x^4 - 3x^2 + 7.$$

Integrate term-by-term:

$$\int 5x^4 dx = x^5; \quad \int -3x^2 dx = -x^3; \quad \int 7 dx = 7x$$

$$\boxed{F(x) = x^5 - x^3 + 7x + C}$$

Practice

Find the general antiderivative of: $f(x) = \frac{4}{\sqrt{x}} - \cos 3x$.

$$\frac{4}{\sqrt{x}} = 4x^{-1/2} \implies \int 4x^{-1/2}dx = 4 \cdot \frac{x^{1/2}}{1/2} = 8\sqrt{x}$$

$$\int -\cos 3x dx = -\frac{1}{3}\sin 3x$$

$$\boxed{F(x) = 8\sqrt{x} - \frac{1}{3}\sin 3x + C}$$

Practice: Initial Condition

Find the antiderivative of $f(x) = 6x^2$ such that $F(2) = 5$.

General antiderivative: $F(x) = 2x^3 + C$
Use condition:

$$F(2) = 2(8) + C = 16 + C$$

$$16 + C = 5 \Rightarrow C = -11$$

$$\boxed{F(x) = 2x^3 - 11}$$

Thank you :)