CALCULUS

Lecture 04

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Tangents and the Derivative: The Big Picture

- Goal: describe how a function changes *at a specific point*.
- Slope of a straight line $\to$ slope of a curve at a point.

Key Limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- Produces a *single number*.
- Existence of the limit is essential.

Slope, Tangent, and Derivative at a Point

Definitions

For $y = f(x)$ at $P(x_0, f(x_0))$:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- Slope of the curve at P.
- Slope of the tangent line at P.
- Tangent line: line through P with slope $f'(x_0)$.
- Measures instantaneous rate of change.

EXAMPLE: TANGENT TO $y = \frac{1}{x}$

Problem

For $y = \dfrac{1}{x}$:

- Find the slope at $x = a \neq 0$.
- Find where the slope equals $-\dfrac{1}{4}$.

Result

$$f'(a) = -\frac{1}{a^2}$$

Behavior of Tangents

- Slope always negative.
- As $a \to 0^{\pm}$, slope $\to -\infty$ (very steep).
- As $|a| \to \infty$, slope $\to 0$ (almost flat).

Physical Meaning and Interpretations

Example: Falling Object

$$y(t) = 16t^2 \quad \text{(distance in feet)}$$

$$v(t) = \frac{dy}{dt} = 32t \quad \Rightarrow \quad v(1) = 32 \text{ ft/sec}$$

The Same Limit Means

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

1. Slope of the graph at x_0.
2. Slope of the tangent line.
3. Instantaneous rate of change.
4. Derivative $f'(x_0)$.

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From Derivative at a Point to a Function

- Earlier: derivative at a fixed point x_0.
- Now: let x vary over the domain.
- Result: a new function $f'(x)$ derived from $f(x)$.

Definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

- Domain of f' may be smaller than domain of f.
- If $f'(x)$ exists, f is *differentiable* at x.

Equivalent Forms and Differentiation

Alternative Difference Quotient

Let $z = x + h$.

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Differentiation

- The process of finding $f'(x)$.
- An operation acting on functions.

Notation

$$f'(x), \quad \frac{dy}{dx}, \quad \frac{df}{dx}, \quad \frac{d}{dx} f(x)$$

Differentiation from the Definition

Example

$$f(x) = \frac{x}{x-1}$$

Method

- Compute $f(x+h)$.
- Form the difference quotient.
- Simplify and take the limit.

Result

$$f'(x) = -\frac{1}{(x-1)^2}.$$

EXAMPLE: $f(x) = \sqrt{x}$ AND ITS TANGENT

Domain: $x > 0$
Derivative

$$f'(x) = \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} = \frac{1}{2\sqrt{x}}.$$

- Requires algebraic manipulation.
- Avoids direct substitution.

Tangent at $x = 4$

$$f'(4) = \frac{1}{4}, \quad (4, 2)$$

$$y - 2 = \frac{1}{4}(x - 4)$$

GRAPHING THE DERIVATIVE

How to Sketch $f'(x)$ from $f(x)$

- Estimate slopes of tangents to $f(x)$.
- Plot points $(x, f'(x))$.
- Connect smoothly.

What the Graph of $f'(x)$ Reveals

- Where f is increasing or decreasing.
- Where rate of change is zero.
- Relative steepness.
- How the rate of change itself varies.

DIFFERENTIABILITY AND ITS LIMITS

Differentiability on Intervals

- Differentiable on (a, b) if $f'(x)$ exists for all x.
- Differentiable on $[a, b]$ if:
  - differentiable on (a, b),
  - right-hand derivative at a,
  - left-hand derivative at b.

One-Sided Derivatives

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

Example: $f(x) = |x|$

- Right derivative at 0: $+1$.
- Left derivative at 0: -1.
- Not differentiable at 0.

Failure of Differentiability

- Corner, cusp,
- vertical tangent,
- discontinuity,
- rapid oscillation.

Theorem

- Differentiability $\Rightarrow$ continuity.
- Converse is false.

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Why Differentiation Rules?

- Derivatives from the definition are exact but slow.
- Common functions follow predictable patterns.
- Rules replace repeated limit computations.

Rules trade foundational rigor for speed and reliability.

Constant and Power Rules

Constant Rule

$$\frac{d}{dx}(c) = 0$$

- Constant functions do not change.
- Horizontal line $\Rightarrow$ zero slope.

Power Rule

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \quad \text{(where defined)}$$

Examples

$$\frac{d}{dx}(x^3) = 3x^2, \quad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}, \quad \frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-7/3}$$

Linearity of Differentiation

Constant Multiple Rule

$$\frac{d}{dx}[cu(x)] = c\frac{du}{dx}$$

Sum and Difference Rules

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

Example

$$y = x^3 + 4x^2 - 5x + 1$$

$$\frac{dy}{dx} = 3x^2 + 8x - 5$$

- Differentiate term by term.
- Polynomials are differentiable everywhere.

Horizontal Tangents

- Horizontal tangent $\iff \dfrac{dy}{dx} = 0.$
- Solve derivative equation to find such points.

Example

$$y = x^4 - 2x^2 + 2$$

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1)$$

Solutions

$$x = -1,\ 0,\ 1$$

PRODUCT RULE

Rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Derivative of a product is not the product of derivatives.

Example

$$y = (x^2 + 1)(x^3 + 3)$$

$$\frac{dy}{dx} = (x^2 + 1)(3x^2) + (x^3 + 3)(2x) = 5x^4 + 3x^2 + 6x$$

Quotient Rule

Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}, \quad v \neq 0$$

Example

$$y = \frac{t^2 - 1}{t^3 + 1}$$

$$\frac{dy}{dt} = \frac{(t^3 + 1)(2t) - (t^2 - 1)(3t^2)}{(t^3 + 1)^2}$$

Higher Derivatives and Summary

Second Derivative

$$f''(x) = \frac{d}{dx}[f'(x)]$$

Example

$$y = x^6, \quad y' = 6x^5, \quad y'' = 30x^4$$

Differentiation Rules

- Constant Rule
- Power Rule
- Constant Multiple Rule
- Sum and Difference Rules
- Product Rule
- Quotient Rule
- Higher-order derivatives

This is the computational core of calculus.

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Why Trigonometric Derivatives?

- Many natural and engineered systems are periodic.
- Oscillations, waves, and vibrations use sine and cosine.
- Their derivatives explain changing motion and forces.

Derivatives of $\sin x$ and $\cos x$

Assumption: x in radians

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x$$

Geometric Interpretation

- $\cos x$ gives slopes of $y = \sin x$.
- $-\sin x$ gives slopes of $y = \cos x$.
- Zero slope at maxima and minima.

DIFFERENTIATION WITH $\sin x$ AND $\cos x$

Examples Differentiate:

- $y = x^2 - \sin x$
- $y = x^2 \sin x$
- $y = \dfrac{\sin x}{x}$
- $y = 5x + \cos x$
- $y = \sin x \cos x$
- $y = \dfrac{\cos x}{1 - \sin x}$

Rules Used

- Sum, Product, and Quotient Rules

DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS

Definitions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

Derivative Formulas

$$\frac{d}{dx}(\tan x) = \sec^2 x, \quad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x, \quad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Key Idea

- All follow from $\sin x$, $\cos x$, and the Quotient Rule.

Higher Derivatives and Summary

Higher-Order Example

$$y = \sec x, \quad y' = \sec x \tan x, \quad y'' = \sec^3 x + \sec x \tan^2 x$$

Differentiability and Continuity

- Trigonometric functions are differentiable on their domains.
- Differentiability implies continuity.
- Limits often evaluated by substitution.

Key Takeaways

- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\cos x) = -\sin x$
- Other trig derivatives follow systematically.
- Trigonometric derivatives model oscillatory motion.

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Why the Chain Rule?

- Many functions are built by combining simpler functions.
- Examples:

$$\sin(x^2), \quad (3x+1)^5, \quad \sqrt{1+x^3}$$

- Earlier rules cannot handle these compositions.

The Chain Rule differentiates composite functions.

Composite Functions and the Chain Rule

Composite Form

$$y = f(g(x))$$

- $g(x)$: inner function
- $f(u)$: outer function

The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- Differentiate outside $\rightarrow$ inside.
- Multiply by derivative of the inner function.

Chain Rule as Rates of Change

- $\dfrac{dy}{du}$: how y changes with u.
- $\dfrac{du}{dx}$: how u changes with x.
- Their product gives how y changes with x.

Dependency Chain

$$x \longrightarrow u \longrightarrow y$$

Chain Rule in Action

Examples

$$(3x+1)^5 \Rightarrow 15(3x+1)^4$$

$$\sin(x^2) \Rightarrow \cos(x^2) \cdot 2x$$

$$\sqrt{1+x^3} \Rightarrow \frac{3x^2}{2\sqrt{1+x^3}}$$

$$\cos(5x^2) \Rightarrow -\sin(5x^2) \cdot 10x$$

General Pattern

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} g'(x)$$

Common Errors and Key Takeaways

Hidden Chain Rule

$$(x^2 + 1)^3 \Rightarrow 3(x^2 + 1)^2 \cdot 2x$$

Common Pitfalls

- Forgetting the inner derivative.
- Differentiating inside and outside incorrectly.
- Treating composites as simple powers.

Summary

- The Chain Rule differentiates composite functions.
- Formula:
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$
- Foundation for implicit differentiation and related rates.

Thank you :)