CALCULUS

Lecture 03

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Intuitive Concept of Continuity

- A function is **continuous** if its graph can be drawn without lifting the pencil.
- It means no holes, breaks, or jumps.
- If a function is continuous at c, the limit as $x \to c$ is simply the function value $f(c)$.

Definition of Continuity at a Point

Definition

Interior Point: A function f is continuous at an interior point c of its domain if:

$$\lim_{x \to c} f(x) = f(c)$$

Endpoint: A function f is continuous at a left endpoint a (or right endpoint b) if:

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b)$$

THE CONTINUITY TEST

A function $f(x)$ is continuous at an interior point $x = c$ if and only if it meets three conditions:

1. $f(c)$ exists (c is in the domain).
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

If any condition fails, f is **discontinuous** at c.

Example: Testing Continuity

Check whether the function is continuous at $x = 1$:

$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 3 & x = 1 \end{cases}$$

- Does $f(1)$ exist?
- Does $\lim_{x \to 1} f(x)$ exist?
- Are they equal?

Conclusion: Is f continuous at $x = 1$?

Types of Discontinuities

- **Removable Discontinuity:** The limit exists, but $f(c)$ is undefined or $f(c) \neq$ limit. (Can "fix" by redefining one point).
- **Jump Discontinuity:** Left-hand and Right-hand limits exist but are different. (Example: Step function).
- **Infinite Discontinuity:** The function goes to $\pm\infty$ at the point. (Example: $1/x^2$ at $x = 0$).
- **Oscillating Discontinuity:** Function oscillates infinitely near the point. (Example: $\sin(1/x)$ at $x = 0$).

Example: Identify the Discontinuity

For each function, identify the type of discontinuity at the given point.

1. $f(x) = \dfrac{x^2 - 1}{x - 1}$ at $x = 1$
2. $f(x) = \begin{cases} 1 & x < 0 \\ 3 & x \geq 0 \end{cases}$ at $x = 0$
3. $f(x) = \dfrac{1}{x}$ at $x = 0$

Hint: Compare left-hand limit, right-hand limit, and function value.

Continuous Functions

Definition

A **continuous function** is one that is continuous at every point of its domain.

- **Polynomials:** Continuous everywhere.
- **Rational Functions:** Continuous wherever they are defined (denominator $\neq 0$).
- **Trigonometric Functions:** Continuous wherever defined ($\sin x, \cos x$ everywhere).
- **Root Functions:** $y = \sqrt[n]{x}$ is continuous on its domain.

Properties of Continuous Functions

If f and g are continuous at $x = c$, then the following are also continuous at $x = c$:

- **Sum/Difference:** $f \pm g$
- **Constant Multiple:** $k \cdot f$
- **Product:** $f \cdot g$
- **Quotient:** f/g (provided $g(c) \neq 0$)
- **Powers/Roots:** $f^n, \sqrt[n]{f}$ (provided defined)

Composites of Continuous Functions

Theorem

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

- **Intuition:** If x is close to c, $f(x)$ is close to $f(c)$, and thus $g(f(x))$ is close to $g(f(c))$.
- **Example:** $y = \sqrt{x^2 - 2x - 5}$ is continuous on its domain because it is a composite of a polynomial and a square root.

Example: Continuity of a Composite

Determine where the function is continuous:

$$f(x) = \sqrt{5 - x^2}$$

- Inner function: $5 - x^2$ (continuous everywhere)
- Outer function: $\sqrt{x}$ (continuous for $x \geq 0$)

Question: For which x is $f(x)$ continuous?

Limits of Continuous Functions

Theorem

If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then:

$$\lim_{x \to c} g(f(x)) = g(b) = g\left(\lim_{x \to c} f(x)\right)$$

- Allows moving the limit inside a continuous function.
- **Example:**

$$\lim_{x \to \pi/2} \cos(2x) = \cos\left(\lim_{x \to \pi/2} 2x\right) = \cos(\pi) = -1$$

Continuous Extension

- Sometimes a function f is undefined at c (hole), but $\lim_{x \to c} f(x) = L$ exists.
- We can define a new function $F(x)$ that is continuous at c by filling the hole:

$$F(x) = \begin{cases} f(x) & x \neq c \\ L & x = c \end{cases}$$

- F is called the **continuous extension** of f to $x = c$.

Example: $f(x) = \frac{x^2+x-6}{x^2-4}$ at $x = 2$.

$$\lim_{x \to 2} f(x) = \frac{5}{4} \implies F(2) = 5/4$$

Example: Continuous Extension

Find k such that the function is continuous at $x = 2$:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

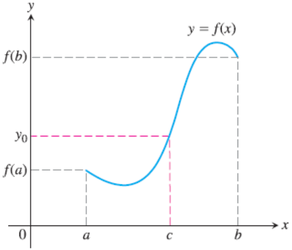
- Compute $\lim_{x \to 2} f(x)$
- Choose k to match the limit

Intermediate Value Theorem (IVT)

Theorem

If f is continuous on a closed interval $[a, b]$, and y_0 is any value between $f(a)$ and $f(b)$, then there exists at least one c in $[a, b]$ such that $f(c) = y_0$.

- **Geometrically:** Any horizontal line $y = y_0$ between $f(a)$ and $f(b)$ must intersect the graph.
- **Key Requirement:** Continuity is essential.



Application: Finding Roots

- **Consequence of IVT:** If f is continuous on $[a, b]$ and changes sign (e.g., $f(a) < 0$ and $f(b) > 0$), then there must be a root c in (a, b) where $f(c) = 0$.

Example: Show $x^3 - x - 1 = 0$ has a root between 1 and 2.

- $f(1) = -1 < 0$
- $f(2) = 5 > 0$
- Since polynomial is continuous, a root exists in $(1, 2)$.

Example: Using IVT

Show that the equation

$$x \cos x = \frac{1}{2}$$

has at least one solution in the interval $(0, 1)$.

- Define $f(x) = x \cos x - \frac{1}{2}$
- Note that f is continuous on $[0, 1]$
- Evaluate $f(0)$ and $f(1)$
- Apply the Intermediate Value Theorem

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Finite Limits as $x \to \pm\infty$

We analyze the behavior of $f(x)$ as x becomes arbitrarily large positive ($x \to \infty$) or large negative ($x \to -\infty$).

Definition

$\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$, there exists an M such that:

$$x > M \implies |f(x) - L| < \epsilon$$

- Intuitively: The graph eventually gets and stays within a narrow band around the line $y = L$.

Basic Limits at Infinity

- **Constant:** $\lim_{x \to \pm\infty} k = k$
- **Reciprocal:** $\lim_{x \to \pm\infty} \frac{1}{x} = 0$

Limit Laws: Standard limit laws (Sum, Product, Quotient) apply to limits at infinity.

Horizontal Asymptotes

Definition

A line $y = b$ is a **horizontal asymptote** of the graph of f if either:

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

- A graph can cross its horizontal asymptote (unlike vertical asymptotes).
- Example: $f(x) = \frac{5x^2+8x-3}{3x^2+2}$ has horizontal asymptote $y = 5/3$.

STRATEGY: LIMITS OF RATIONAL FUNCTIONS

To find $\lim_{x \to \infty} \frac{P(x)}{Q(x)}$:

1. Divide numerator and denominator by the *highest power of x* in the denominator.
2. Use the fact that terms like $k/x^n \to 0$.

Example:

$$\lim_{x \to \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x \to \infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}} = \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$

Example: Limits at Infinity

Evaluate:

$$\lim_{x \to \infty} \frac{3x^3 - 5x + 1}{2x^3 + x^2}$$

- Identify dominant powers
- Divide numerator and denominator by x^3
- Simplify using limits

Oblique (Slant) Asymptotes

- Occur when degree of numerator is exactly **one greater** than degree of denominator.
- **Method:** Use polynomial long division.

$$f(x) = \text{Linear Part } (mx + b) + \text{Remainder } R(x)$$

- Since $\lim_{x\to\infty} R(x) = 0$, the graph approaches the line $y = mx + b$.

Example: $f(x) = \frac{x^2-3}{2x-4} = \frac{x}{2} + 1 + \frac{1}{2x-4}$.
Asymptote is $y = \frac{x}{2} + 1$.

Infinite Limits

Describes behavior where function values grow without bound.

- $\lim_{x\to c} f(x) = \infty$: Values become arbitrarily large positive.
- $\lim_{x\to c} f(x) = -\infty$: Values become arbitrarily large negative.

Important: The limit *does not exist* in the numeric sense. The symbol ∞ describes the behavior.

Example:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

Vertical Asymptotes

Definition

A line $x = a$ is a **vertical asymptote** of the graph of f if either:

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

- Typically occur at zeros of the denominator in rational functions (where numerator is non-zero).
- **Example:** $f(x) = \frac{x+3}{x+2}$. Vertical asymptote at $x = -2$.

Example: Vertical Asymptotes

Find all vertical asymptotes of:

$$f(x) = \frac{2x+1}{(x-1)(x+2)}$$

- Identify where denominator is zero
- Check behavior from left and right

Dominant Terms

When analyzing functions for large $|x|$ (end behavior) or near singularities:

- **At Infinity:** The term with the highest power dominates.
  $3x^4 - 2x^3 + ... \approx 3x^4$ as $x \to \infty$.
- **Near Zero:** The term with the lowest power (or highest negative power) dominates.
  $\frac{1}{x} + x \approx \frac{1}{x}$ as $x \to 0$.

Thank you :)