CALCULUS

Lecture 02

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Introduction to Rates of Change

- Calculus is a tool to help us understand how functional relationships change.
- Examples include the position or speed of a moving object as a function of time, or the changing slope of a curve.
- The concept of a limit is fundamental to finding the velocity of a moving object and the tangent to a curve.
- We begin with an informal approach using average and instantaneous rates of change.

Average Speed

- **Galileo's Law of Free Fall**: A solid object dropped from rest near the earth's surface falls a distance proportional to the square of the time.
- Equation: $y = 16t^2$ (distance in feet, time in seconds).
- **Average Speed**: Found by dividing the distance covered by the time elapsed.
- Formula: $\frac{\Delta y}{\Delta t} = \frac{\text{change in distance}}{\text{change in time}}$.

Example: Average Speed Calculation

Example: A rock breaks loose from a tall cliff.

- Average speed during the first 2 seconds:

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32 \text{ ft/sec}$$

- Average speed between second 1 and second 2:

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48 \text{ ft/sec}$$

INSTANTANEOUS SPEED

- To find speed at a single instant t_0, we calculate average speed over shorter and shorter intervals $[t_0, t_0 + h]$.
- Average speed formula:

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$$

- As h approaches 0, the average speed approaches a limiting value, which is the instantaneous speed.
- For $t_0 = 1$: Average speed $= 32 + 16h$. As $h \to 0$, speed $\to 32$ ft/sec.

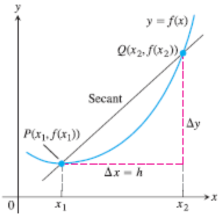
Average Rate of Change

Definition

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is:

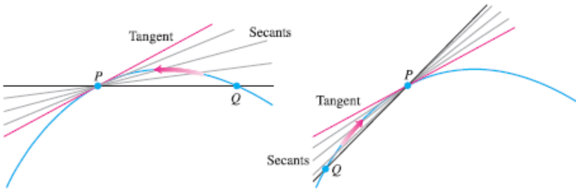
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0$$

Geometrically, this is the slope of the secant line through points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$.

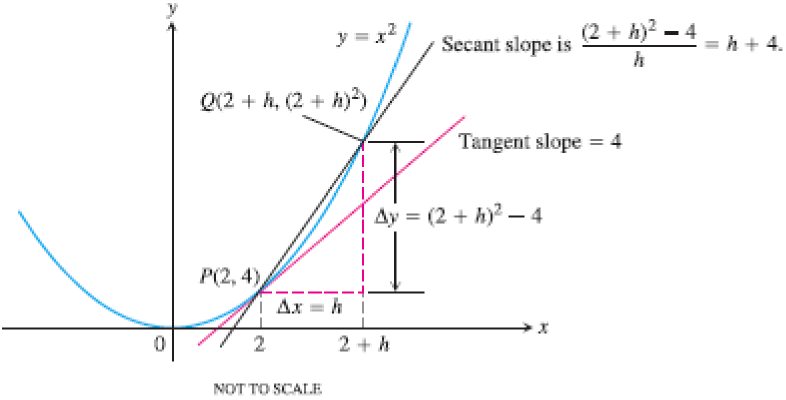


Defining the Slope of a Curve

- We define the tangent to a curve at a point P by analyzing the behavior of secants through P and nearby points Q.
- **Procedure**:
  1. Calculate the slope of the secant PQ.
  2. Investigate the limit of the secant slope as $Q \to P$.
  3. If the limit exists, it is the slope of the curve at P and the slope of the tangent line.



Example: Slope of a Parabola



EXAMPLE: SLOPE OF A PARABOLA

Example: Find the slope of $y = x^2$ at $P(2, 4)$.

- Secant slope through $P(2, 4)$ and $Q(2 + h, (2 + h)^2)$:

$$\frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h}{h} = h + 4$$

- As $Q \to P$ along the curve, $h \to 0$.
- The secant slope $h + 4$ approaches 4.
- The tangent equation is $y - 4 = 4(x - 2)$ or $y = 4x - 4$.

Instantaneous Rates and Tangents

- The instantaneous rate of change corresponds to the slope of the tangent line.
- The average rate of change corresponds to the slope of a secant line.
- This connection is fundamental to calculus and is investigated via the limit concept.

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Informal Definition of a Limit

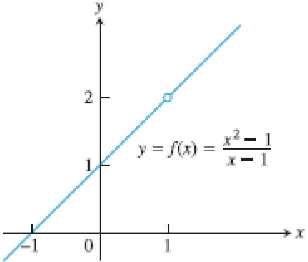
- We analyze function behavior near a point x_0, even if the function is undefined at x_0.
- **Definition**: If $f(x)$ is arbitrarily close to L for all x sufficiently close to x_0, we say f approaches the limit L as x approaches x_0:
$$\lim_{x \to x_0} f(x) = L$$
- This means values of $f(x)$ are close to L whenever x is close to x_0.

Example

Behavior of $f(x) = \frac{x^2-1}{x-1}$ near $x = 1$.

- Defined for all $x \neq 1$
- $f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1$ for $x \neq 1$
- As $x \to 1$, $f(x) \to 2$

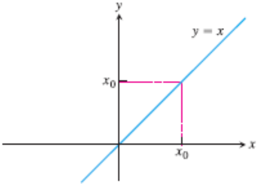
$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$



Identity and Constant Functions

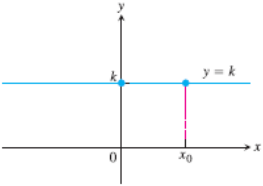
- **Identity Function** $(f(x) = x)$:

$$\lim_{x \to x_0} x = x_0$$



- **Constant Function** $(f(x) = k)$:

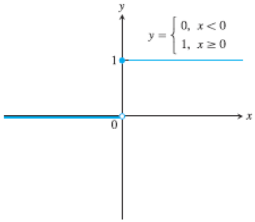
$$\lim_{x \to x_0} k = k$$



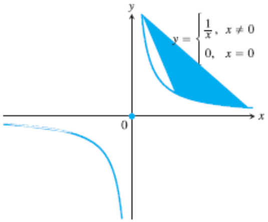
When Limits Fail to Exist

A limit $\lim_{x \to 0} f(x)$ does not exist if:

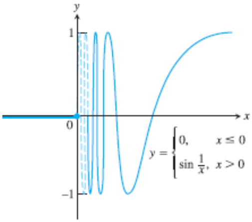
1. **Jump**: The function jumps values at the point (e.g., Unit step function).
2. **Grow too large**: Values grow arbitrarily large (e.g., $1/x$).
3. **Oscillate**: Values oscillate too much (e.g., $\sin(1/x)$).



(a) Unit step function $U(x)$



(b) $g(x)$



(c) $f(x)$

Theorem: Limit Laws

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$:

- **Sum**: $\lim(f + g) = L + M$.
- **Difference**: $\lim(f - g) = L - M$.
- **Constant Multiple**: $\lim(kf) = kL$.
- **Product**: $\lim(f \cdot g) = L \cdot M$.
- **Quotient**: $\lim \frac{f}{g} = \frac{L}{M}, \quad M \neq 0$.
- **Power/Root**: $\lim[f(x)]^n = L^n$ and $\lim \sqrt[n]{f(x)} = \sqrt[n]{L}$.

Example: Find the limits

(a) $\lim\limits_{x \to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim\limits_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim\limits_{x \to -2} \sqrt{4x^2 - 3}$

Solution

(a) $\lim\limits_{x \to c}(x^3 + 4x^2 - 3) = \lim\limits_{x \to c} x^3 + 4 \lim\limits_{x \to c} x^2 - 3 = c^3 + 4c^2 - 3$

$$\textbf{(b)} \lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$$

$$\textbf{(c)} \lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{4(-2)^2 - 3} = \sqrt{16 - 3} = \sqrt{13}$$

Polynomials and Rational Functions

- **Theorem (Polynomials)**: If $P(x)$ is a polynomial, then:

$$\lim_{x \to c} P(x) = P(c)$$

- **Theorem (Rational Functions)**: If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$:

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

- Method: Substitution.

Eliminating Zero Denominators Algebraically

If substitution leads to $0/0$, cancel common factors.

Example: Evaluate $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x}$.

- Numerator and denominator are zero at $x = 1$.
- Factor and cancel $(x-1)$:

$$\frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}, \quad x \neq 1$$

- Limit calculation:

$$\lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$$

The Sandwich Theorem

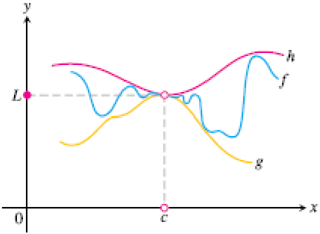
Theorem (Sandwich Theorem)

Suppose $g(x) \leq f(x) \leq h(x)$ for all x near c (except possibly at c). If

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

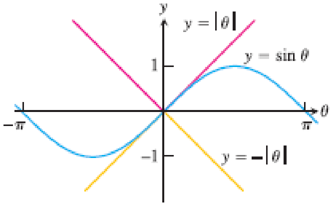
then

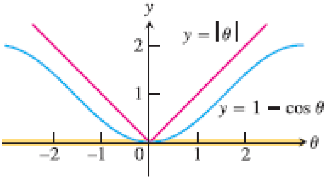
$$\lim_{x \to c} f(x) = L.$$



APPLICATIONS OF SANDWICH THEOREM

- **Limit of Sine**: $\lim_{\theta \to 0} \sin\theta = 0$.
- **Limit of Cosine**: $\lim_{\theta \to 0} \cos\theta = 1$.
- **Absolute Value**: If $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.





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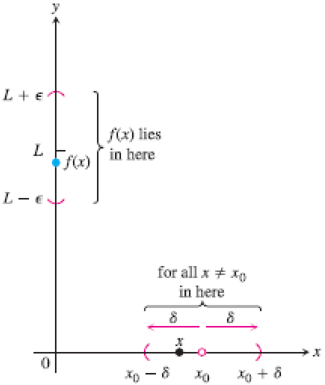
Precise Definition of a Limit

Definition

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0. We say $\lim_{x \to x_0} f(x) = L$ if: For every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x:

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon$$

- ϵ (epsilon): Error tolerance (output).
- δ (delta): Distance from x_0 (input).



Example: Verifying a Limit

Show $\lim_{x \to 1}(5x - 3) = 2$.

- Set $x_0 = 1$, $f(x) = 5x - 3$, $L = 2$.
- We want $|f(x) - 2| < \epsilon$ whenever $0 < |x - 1| < \delta$.
- Work backward:

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < \epsilon$$

$$|x - 1| < \epsilon/5$$

- Choose $\delta = \epsilon/5$. This guarantees the condition is met.

Finding Deltas Algebraically

Steps to find δ for a given ϵ:

1. Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0.
2. Find a value $\delta > 0$ that places the interval $(x_0 - \delta, x_0 + \delta)$ inside (a, b).

Example: $\lim_{x \to 5} \sqrt{x - 1} = 2$, find δ for $\epsilon = 1$.

- Solve $|\sqrt{x - 1} - 2| < 1 \Rightarrow 1 < \sqrt{x - 1} < 3$.
- Square: $1 < x - 1 < 9 \Rightarrow 2 < x < 10$.
- Interval is $(2, 10)$. Center is $x_0 = 5$.
- Distance to endpoints: $5 - 2 = 3$ and $10 - 5 = 5$.
- Choose $\delta = 3$ (minimum distance).

Using the Definition to Prove Theorems

- The precise definition is used to prove general limit laws.
- **Example**: Proof of Sum Rule $\lim(f+g) = L + M$.
- Uses the Triangle Inequality: $|a+b| \leq |a| + |b|$.
- Choose $\delta = \min(\delta_1, \delta_2)$ to satisfy conditions for both f and g simultaneously.

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Introduction to One-Sided Limits

- **Context:** Sometimes a function $f(x)$ behaves differently depending on whether x approaches a value c from the left or the right.
- **Ordinary (Two-Sided) Limit:** Requires $f(x)$ to approach the same value L from *both* sides.
- **One-Sided Limit:** Considers the limit strictly from one side (left or right).

Right-Hand and Left-Hand Limits

Definition (Right-Hand Limit)

The limit of $f(x)$ as x approaches c from the right is L, written

$$\lim_{x \to c^+} f(x) = L$$

if $f(x)$ gets arbitrarily close to L for all $x > c$ sufficiently close to c.

Definition (Left-Hand Limit)

The limit of $f(x)$ as x approaches c from the left is M, written

$$\lim_{x \to c^-} f(x) = M$$

if $f(x)$ gets arbitrarily close to M for all $x < c$ sufficiently close to c.

Example: The Sign Function

Consider the function $f(x) = \frac{x}{|x|}$.

- Defined for all $x \neq 0$.
- For $x > 0$, $f(x) = 1$. Thus:

$$\lim_{x \to 0^+} f(x) = 1$$

- For $x < 0$, $f(x) = -1$. Thus:

$$\lim_{x \to 0^-} f(x) = -1$$

- Since $1 \neq -1$, the two-sided limit $\lim_{x \to 0} f(x)$ does not exist.

Example: Semicircle

Consider $f(x) = \sqrt{4 - x^2}$ on the domain $[-2, 2]$.

- At $x = -2$: Only the right-hand limit exists.

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$$

- At $x = 2$: Only the left-hand limit exists.

$$\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$$

- At these endpoints, the ordinary two-sided limits do not exist.

Relation Between One-Sided and Two-Sided Limits

Theorem

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \quad \textit{and} \quad \lim_{x \to c^+} f(x) = L$$

- This theorem is a primary tool for determining limit existence at piecewise boundaries.

Precise Definition of One-Sided Limits

Formal Definition (Right-Hand Limit)

$\lim_{x \to x_0^+} f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$x_0 < x < x_0 + \delta \implies |f(x) - L| < \epsilon$$

Formal Definition (Left-Hand Limit)

$\lim_{x \to x_0^-} f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$x_0 - \delta < x < x_0 \implies |f(x) - L| < \epsilon$$

Limits Involving $(\sin\theta)/\theta$

Theorem

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad (\theta \textit{ in radians})$$

- Confirmed geometrically and by the Sandwich Theorem.
- Crucial for finding derivatives of trigonometric functions.
- **Note:** Since $\frac{\sin\theta}{\theta}$ is an even function, the left and right limits are equal.

Examples: Trigonometric Limits

Evaluate $\lim_{h \to 0} \frac{\cos h - 1}{h}$.

- Use half-angle identity: $\cos h = 1 - 2\sin^2(h/2)$.
- Result: limit is 0.

Evaluate $\lim_{x \to 0} \frac{\sin 2x}{5x}$.

- Manipulate to match the form $\frac{\sin u}{u}$.

$$\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} = \frac{2}{5}(1) = \frac{2}{5}$$

Thank you :)