Calculus

Lecture 01

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Functions

Definition

A **function** f from a set D to a set Y is a rule that assigns a unique element $f(x) \in Y$ to each element $x \in D$.

- **Domain (D):** The set of all possible input values.
- **Range:** The set of all values $f(x)$ as x varies throughout D.
- **Dependent Variable:** usually y (output).
- **Independent Variable:** usually x (input).

Natural Domain

If the domain is not explicitly stated, it is assumed to be the largest set of real x-values for which the formula gives real y-values.

Representing Functions

Functions can be represented in four ways:

1. **Algebraically:** By a formula (e.g., $A = \pi r^2$).
2. **Visually:** By a graph.
3. **Numerically:** By a table of values.
4. **Verbally:** By a description in words.

Vertical Line Test

- A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

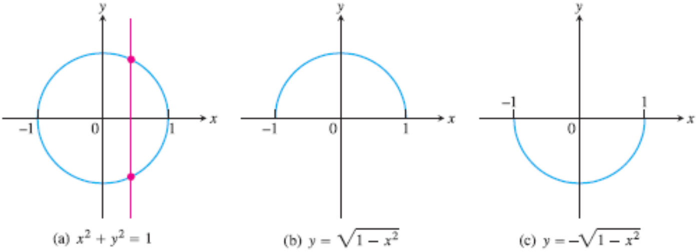


FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

Domain and Range

Example

Verify the natural domains and ranges:

1. $y = x^2$
   - Domain: $(-\infty, \infty)$
   - Range: $[0, \infty)$
2. $y = \sqrt{1 - x^2}$
   - Domain: $[-1, 1]$ (since $1 - x^2 \geq 0$)
   - Range: $[0, 1]$
3. $y = 1/x$
   - Domain: $x \neq 0$
   - Range: $y \neq 0$

Piecewise-Defined Functions

Functions described by different formulas on different parts of their domain.

Example (Absolute Value Function)

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

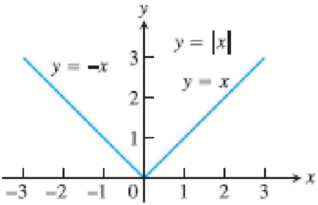


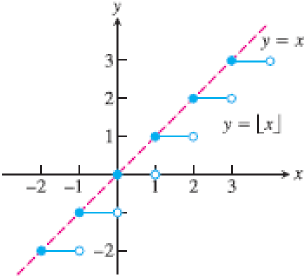
FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Piecewise-Defined Functions

Example (Greatest Integer (Floor) Function)

$y = \lfloor x \rfloor$: The greatest integer less than or equal to x.

- $\lfloor 2.4 \rfloor = 2$
- $\lfloor -0.3 \rfloor = -1$



Increasing and Decreasing Functions

Let f be defined on an interval I, and let $x_1, x_2 \in I$ with $x_1 < x_2$.

Definition

- **Increasing:** If $f(x_2) > f(x_1)$ for all $x_1 < x_2$.
- **Decreasing:** If $f(x_2) < f(x_1)$ for all $x_1 < x_2$.

Example

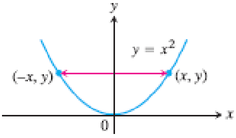
$y = x^2$ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

Even and Odd Functions (Symmetry)

Even Function

$f(-x) = f(x)$ for every x.

- Graph symmetric about the y-axis
- Example: $y = x^2, \ \cos x$

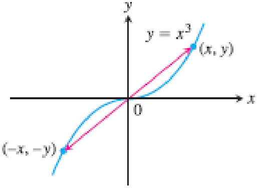


(a)

Odd Function

$f(-x) = -f(x)$ for every x.

- Graph symmetric about the origin
- Example: $y = x^3, \ \sin x$



(b)

Common Functions

- **Linear Functions:** $f(x) = mx + b$.
- **Power Functions:** $f(x) = x^a$.
  - $a = n$ (positive integer): $x, x^2, x^3 \dots$
  - $a = -1, -2$: Hyperbolas like $1/x$.
  - $a = 1/2, 1/3$: Root functions $\sqrt{x}$, $\sqrt[3]{x}$.
- **Polynomials:** $p(x) = a_n x^n + \cdots + a_0$. Domain is $(-\infty, \infty)$.
- **Rational Functions:** Ratio of two polynomials $p(x)/q(x)$. Domain excludes $q(x) = 0$.
- **Exponential Functions:** $f(x) = a^x$ $(a > 0, a \neq 1)$.
- **Logarithmic Functions:** $f(x) = \log_a x$ (Inverse of exponential).

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Algebraic Combinations of Functions

Let f and g be functions with domains $D(f)$ and $D(g)$. For $x \in D(f) \cap D(g)$:

- **Sum:** $(f+g)(x) = f(x) + g(x)$
- **Difference:** $(f-g)(x) = f(x) - g(x)$
- **Product:** $(fg)(x) = f(x)g(x)$
- **Quotient:** $(f/g)(x) = f(x)/g(x)$ (where $g(x) \neq 0$)

Note

The domain of the arithmetic combination is the intersection of the domains, excluding points where the denominator is zero for quotients.

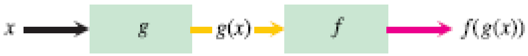
Composition of Functions

Definition

The **composite function** $f \circ g$ is defined by:

$$(f \circ g)(x) = f(g(x))$$

Domain: The set of all x in the domain of g such that $g(x)$ lies in the domain of f.



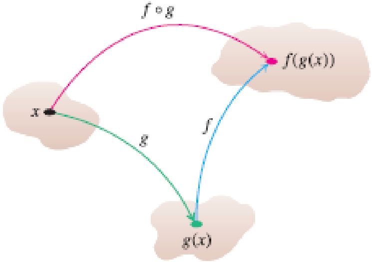
Composition of Functions

Example

If $f(x) = \sqrt{x}$ and $g(x) = x + 1$:

$$(f \circ g)(x) = \sqrt{x+1}$$

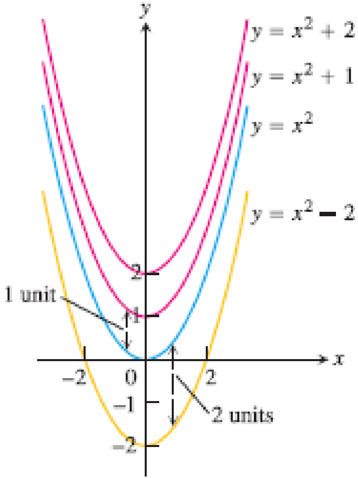
Domain is $[-1, \infty)$.



Shifting Graphs (Translations)

Vertical Shifts:
Let $k > 0$.

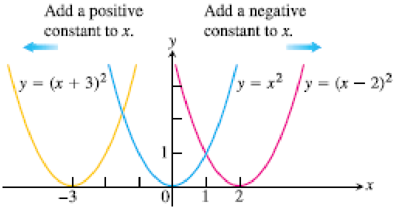
- $y = f(x) + k$: Shift **up** k units.
- $y = f(x) - k$: Shift **down** k units.

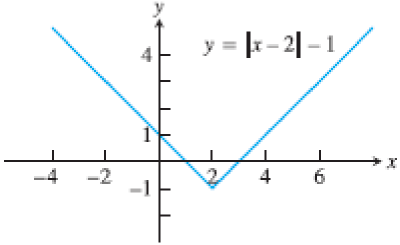


Shifting Graphs (Translations)

Horizontal Shifts: Let $h > 0$.

- $y = f(x + h)$: Shift **left** h units.
- $y = f(x - h)$: Shift **right** h units.

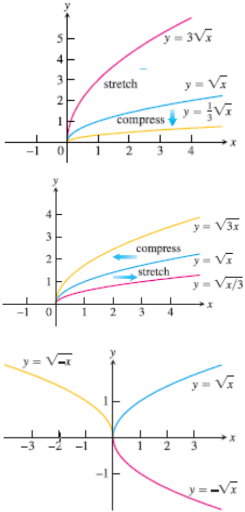




Scaling and Reflecting Graphs

Let $c > 1$.

- **Vertical Scaling:**
  - $y = cf(x)$: Stretch vertically by factor c.
  - $y = \frac{1}{c}f(x)$: Compress vertically by factor c.
- **Horizontal Scaling:**
  - $y = f(cx)$: Compress horizontally by factor c.
  - $y = f(x/c)$: Stretch horizontally by factor c.
- **Reflections:**
  - $y = -f(x)$: Reflect across **x-axis.**
  - $y = f(-x)$: Reflect across **y-axis.**



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Angles and Radians

- **Radian Measure:** Defined by arc length s on a circle of radius r:

$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta$$

- On a unit circle ($r = 1$), the angle in radians equals the arc length.
- **Conversion:**

$$\pi \text{ radians} = 180^\circ$$

$$1 \text{ rad} \approx 57.3^\circ, \quad 1^\circ \approx 0.017 \text{ rad}$$

Convention: In calculus, always assume angles are in **radians** unless degrees are explicitly stated.

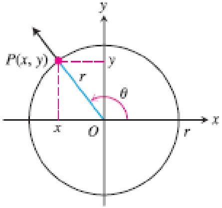
The Six Basic Trigonometric Functions

Defined via point $P(x, y)$ on a circle of radius r:

- $\sin\theta = \frac{y}{r}$
- $\cos\theta = \frac{x}{r}$
- $\tan\theta = \frac{y}{x}$
- $\csc\theta = \frac{r}{y} = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{r}{x} = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{x}{y} = \frac{1}{\tan\theta}$

Periodicity:

- sin, cos, sec, csc have period 2π.
- tan, cot have period π.



Key Trigonometric Identities

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Symmetry (Even/Odd)

- Even: $\cos(-x) = \cos x$, $\sec(-x) = \sec x$
- Odd: $\sin(-x) = -\sin x$, $\tan(-x) = -\tan x$

Addition and Double-Angle Formulas

Addition Formulas:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Double-Angle Formulas:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Half-Angle Formulas (Useful for Integration):

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Law of Cosines & Inequalities

Law of Cosines

For a triangle with sides a, b, c and angle θ opposite c:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Two Special Inequalities

For any angle θ in radians:

$$-|\theta| \leq \sin\theta \leq |\theta|$$

$$-|\theta| \leq 1 - \cos\theta \leq |\theta|$$

These are crucial for proving continuity and limits of trig functions.

Thank you :)